Differential Equations

1 Mark questions:

1. Define a differential equation.
   It is an equation containing derivatives.
2. Define order of a differential equation.
   It is highest order of derivative appearing in the given equation.
3. Define degree of a differential equation.
   It is the highest power of highest ordered derivative appearing in the given equation.
4. Define general solution of a differential equation.
   It is solution of given differential equation and it contains arbitrary constants.
5. Define particular solution of a differential equation.
   It is that solution of given differential equation and is free from arbitrary constants.

2 Marks questions:

1. Form a differential equation of family of
   i. Straight lines with slope = m and passing through origin.
      Consider a straight line with slope = m and passing through origin
      i.e. y = mx -----(1)
      $\Rightarrow y^1 = m$ -------(2) $\quad\therefore\ y = x\,y^1$
   ii. Circles with centre on y-axis and passing through origin.
      $Consider\ x^2+y^2+2fy=0 ---(1) \Rightarrow 2x+2yy^1+2fy^1=0$
      $\div(2) \Rightarrow -\left(\frac{x+yy^1}{y^1}\right)=f \qquad \Rightarrow x^2+y^2-2\left(x+yy^1\right)=0$
2. Solve the following by using separation of variables.
   i. $x\,dy+y\,dx=dx+dy$
      $\Rightarrow d(xy)=d(x+y) \qquad \therefore \int d(xy)=\int d(x+y) \Rightarrow xy=x+y+c$
   ii. $\frac{dy}{dx}+\frac{\sqrt{a-y^2}}{\sqrt{1-x^2}}=0$
      $\Rightarrow \int \frac{dy}{\sqrt{1-y^2}}+\int \frac{dx}{\sqrt{1-x^2}}=0 \Rightarrow \sin^{-1}y+\sin^{-1}x=c$
   iii. $\left(1+x^2\right)\frac{dy}{dx}+\left(1+y^2\right)=0$

$$\Rightarrow \int \frac{dy}{\sqrt{1+y^2}} + \int \frac{dx}{\sqrt{1+x^2}} = 0 \ \Rightarrow \tan^{-1} y + \tan^{-1} x = c$$

iv. $y_1 = (1+x).(1+y^2)$

$$\Rightarrow \int \frac{dy}{1+y^2} + \int (1+x)dx \Rightarrow \tan^{-1} y = x + \frac{x^2}{2} + c$$

v. $(2y-1)dx + (2x+3)dy = 0$

$$\Rightarrow \int \frac{dx}{2x+3} + \int \frac{dy}{2y-1} = 0 \Rightarrow \frac{1}{2}\log(2x+3) + \frac{1}{2}\log(2y-1) = c$$

vi. $x^2 \frac{dy}{dx} = 1 + y^2$

$$\Rightarrow \int \frac{dx}{1+y^2} + \int \frac{dx}{x^2} = 0 \Rightarrow \tan^{-1} y = -\frac{1}{x} + c$$

vii. $\frac{dy}{dx} = 3x^2 + 2$

$$\Rightarrow \int dy = \int (3x^2+2)dx \Rightarrow y = \frac{3x^2}{3} + 2x \Rightarrow x^3 + 2x + c$$

viii. $y_1 = e^x + 1$

$$\Rightarrow \int dy = \int (e^x+1)dx \Rightarrow y = e^x + x + c$$

ix. $\frac{dx}{x} + \frac{dy}{y} = 0$

$$\Rightarrow \int \frac{dx}{x} + \int \frac{dy}{y} = 0 \ \Rightarrow \log x + \log y = c$$

x. $x\,dy - y\,dy = 0$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = 0 \ \Rightarrow \int d(y/x) = 0 \ \therefore \frac{y}{x} = c$$

xi. $\cos x\,dx - \sin y\,dy = 0$

$$\Rightarrow \int \cos x\,dx + \int \sin y\,dy = 0 \Rightarrow \sin x - \cos y = c$$

xii. $y\cos^2 x\,dy + dx = 0$

$$\Rightarrow \int y\,dy + \int \sec^2 x\,dx = 0 \Rightarrow \frac{y^2}{2} + \tan y = c$$

xiii. $\frac{e^x\,dx}{1+e^x} + \frac{e^y dy}{1+e^y} = 0$

$$\Rightarrow \int \frac{d(1+e^x)}{1+e^x} + \int \frac{d(1+e^y)}{1+e^y} = 0 \Rightarrow \log(1+e^x) + \log(1+e^y) = c$$

xiv. $y(1+x^2)dy + x(1+y^2)dy = 0$

$$\Rightarrow \int \frac{y}{1+y^2}dy + \int \left(x/\left(1+x^2\right)\right)dx = 0 \Rightarrow \int \frac{2y\ dy}{1+y^2} + \int \frac{2y\ dx}{1+x^2} = 0$$

$$\Rightarrow \log\left(1+y^2\right) + \log\left(1+x^2\right) = c$$

xv. $\frac{dy}{dx} = e^{x-y}$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x}{e^y} \quad \therefore \int e^y dy + \int e^x\ dx \Rightarrow e^y = e^x + c$$

Problems on homogenous equations 3 mark question:

1. $Solve\ \frac{dy}{dx} = \frac{x+y}{x-y}$

$$Given,\ \frac{dy}{dx} = \frac{x+y}{x-y} ---(1),\ \text{put}\, \text{y} = \text{V}x$$

$$\therefore \frac{dy}{dx} = V + x\frac{dV}{dx} \qquad \therefore (1) \Rightarrow V + x\frac{dV}{dx} = \frac{x+Vx}{x-Vx} = \frac{1+V}{1-V}$$

$$\therefore x\frac{dV}{dx} = \frac{1+V}{1-V} - V = \frac{1+V^2}{1-V} \qquad \int \frac{(1-V)dV}{\left(1+V^2\right)} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dV}{1+V^2} - \frac{1}{2}\int \frac{V.2dV}{1+V^2} = \log x \Rightarrow \tan^{-1}(y/x) - \log \frac{\sqrt{x^2+y^2}}{x} = \log x$$

$$\Rightarrow \tan^{-1} y/x - \log \sqrt{x^2+y^2} + c$$

<u>Linear Differential Equations:</u>

<u>Solve the following:</u>

1. $\frac{dx}{dy} - \frac{x}{y} = 2y$

$$P = \frac{-1}{y} \quad ; \quad Q = 2y$$

$$\int p\,dy = \int \frac{-1}{y}dy = -\log y = \log\left(y^{-1}\right)$$

$$\therefore I.f. = e\int Pdy = e^{\log\left(y^{-1}\right)} = y^{-1}$$

$$\therefore Solution\ \ is\ \ xy^{-1} = \int y^{-1} 2y\,dy = 2\int dy = 2y \Rightarrow \frac{x}{y} = 2y + c$$

2. $xy_1 + 2y = x^2$

$$\Rightarrow y_1+\frac{2y}{x}=x \qquad P=\frac{2}{x} \quad ; \quad Q=x$$

$$\int Pdx=2\log x=\log\left(x^2\right) \qquad I.f \text{ is } e^{\int Pdx}=e^{\log\left(x^2\right)}=x^2$$

$$\therefore Solution \text{ is } y.x^2=\int x^2.x\,dx=\int x^3dx=\frac{x^4}{4} \quad \therefore yx^2=\frac{x^4}{4}+c$$

3. $y_1+y\cot x=2x+x^2\cot x \quad ; \quad y(\pi/2)=0$

$$P=\cot x \quad ; \quad Q=2x+x^2\cot x$$

$$\int Pdx=\int \cot x\,dx=\log \sin x \qquad \therefore e^{\int Pdx}=e^{\log\sin x}=\sin x$$

$$\therefore Solution \text{ is } y.\sin x=\int \left(2x\sin x+x^2\cot x\sin x\right)dx$$

$$=2\int x\sin x\,dx+\int x^2\cos x\,dx=2\int x\sin x\,dx+\int x^2d\left(\sin x\right)$$

$$=2\int x\sin x\,dx+x^2\sin x-\int 2x\sin x\,dx \qquad \therefore y\sin x=x^2\sin x+c$$

$$given, y\left(\pi/2\right)=0 \quad \therefore Q=\pi^2+c \quad \therefore c=\pi^2 \quad \therefore y\sin x=x^2\sin x-\pi^2$$

4. $y_1+3y=e^{-2x}$

$$P=3 \quad ; \quad Q=e^{-2x}dx$$

$$\int Pdx=\int 3dx=3x \quad I.f.=e^{\int Pdx}=e^{3x}$$

$$\therefore Solution \text{ is } y.e^{3x}=\int e^{3x}e^{-2x}\,dx \quad =\int e^xdx=e^x \quad \therefore y.e^{3x}=e^x+c$$

5. $\left(x+3y^2\right)y_1=y\ , \quad y>0\Rightarrow y_1=\dfrac{y}{x+3y^2}$

$$\therefore \frac{dx}{dy}=\frac{x+3y^2}{y} \qquad \therefore \frac{dx}{dy}-\frac{x}{y}=3y$$

$$P=-\frac{1}{y} \quad ; \quad Q=3y \qquad \therefore \int Pdy=-\log y=\log\left(y^{-1}\right)$$

$$I.f.=e^{\log\left(y^{-1}\right)}=\frac{1}{y}$$

$$\therefore Solution \text{ is } x.\frac{1}{y}=\int \frac{1}{y}.3y\ dy=3 \quad \therefore \frac{x}{y}=3y+c$$

6. $y_1+2y\tan x=\sin x$

$$P=2\tan x \quad ; \quad Q=\sin x$$

$$\int Pdx=\int 2\tan x\ dx=2\log\cos x=\log\left(\cos x\right)^2$$

$$I.f.=e^{\log\left(\cos^2 x\right)}=\cos^2 x$$

$$\therefore Solution \text{ is } y.\cos^2 x=\int \sin x.\cos^2 x\ dx=$$

$$=-\int \left(\cos x\right)^2\ d\left(\cos x\right) \quad \therefore y\cos^2 x=-\cos^3 x+c$$

7. $(x+y)\frac{dy}{dx}=1 \quad \Rightarrow \frac{dx}{dy}=x+y$

$\Rightarrow \frac{dx}{dy}-x=y \quad \therefore P=1 \quad ; \; Q=x+y$

$\int Pdy=\int -1dy=-y \qquad I.f.=e^{\int Pdy}=e^{-y}$

$Sol.\ is\ \ x.e^{-y}=\int e^{-y}.y\,dy=\int y\,d\left(-e^{-y}\right)$

$=-ye^{-y}-\int -e^{-y}=-ye^{-y}-e^{-y} \quad \therefore x e^{-y}+e^{-y}(y+1)=c$

8. $\left(1+x^2\right)y_1+2xy=\frac{1}{\left(1+x^2\right)}$

$\therefore y_1+\frac{2x}{1+x^2}y=\frac{1}{\left(1+x^2\right)} \quad \therefore P=\frac{2x}{1+x^2} \quad ; \; Q=\frac{1}{\left(1+x^2\right)^2}$

$\int Pdx=\int \frac{2x}{1+x^2}=\log\left(1+x^2\right) \qquad I.f.=e^{\int Pdx}=e^{\log\left(1+x^2\right)}=\left(1+x^2\right)$

$Sol.\ is\ \ y.\left(1+x^2\right)=\int \left(1+x^2\right).\frac{1}{\left(1+x^2\right)}dx=\int \frac{1}{1+x^2}\,dx$

$\therefore y\left(1+x^2\right)=\tan^{-1}x+c$

9. $y_1-3y\cot x=\sin 2x$

$\therefore P=-3\cot x \quad ; \; Q=\sin 2x$

$\int Pdx=-3\int \cot x=-3\log\sin x=\log\left(\sin^{-3}x\right)$

$I.f.=e^{\int Pdx}=e^{\log(\sin x)^{-3}}=(\sin x)^{-3}$

$Sol.\ is\ \ y.(\sin x)^{-3}=\int \left(\sin^{-1}x\right)^{-1}.2\sin x\cos x\,dx$

$=\int \frac{\cos x\ dx}{\sin^2 x}=\ 2\int \sec x\tan x\ dx=2\sec x$

10. $\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$

$\therefore P=\frac{1}{1+y^2} \quad ; \; Q=\frac{\tan^{-1}y}{a+y^2}$

$\int Pdy=\int \frac{1}{1+y^2}dy=\tan^{-1}y \quad \therefore I.f.=e^{\int Pdx}=e^{\tan^{-1}y}$

$Sol.\ is\ \ x.e^{\tan^{-1}y}=\int e^{\tan^{-1}y}\frac{\tan^{-1}y}{1+y^2}dy=\int t\,e^t\ dt \;,\; t=\tan^{-1}y$

$=\int t\,d\left(e^t\right)=te^t-\int e^t dt=te^t-e^t \quad \therefore xe^{\tan^{-1}y}=\tan^{-1}y-e^{\tan^{-1}y}+c$

Statement problems:

1. Find equation of a curve which passes through origin given that slope at any point on it = sum of coordinates.

$$given\ \frac{dy}{dx} = x + y \qquad \& \qquad y(0) = 0$$

$$\frac{dy}{dx} - y = x \qquad \therefore P = -1 \qquad ; \qquad Q = x$$

$$\therefore \int Pdx = \int -1dx = -x \ \therefore\ I.f. = e^{\int Pdx} = e^{-x}$$

$$Sol.\ is\ y\ .\ e^{-x}.x\,dx \qquad \therefore ye^{-x} = \int x\,d\left(-e^{-x}\right)$$

$$= -xe^{-x} - \int e^{-x}dx \qquad ye^{-x} = -xe^{-x} - e^{-x} + c$$

$$given\ y(0) = 0\ \therefore 0 = 0 - 1 + c\ \therefore c = 1 \therefore ye^{-x} = -(x+1)e^{-x} + 1$$

2. Find the equation of a curve which passes through (0, 2) given that sum of coordinates at any point exceeds slope at that point by 5.

$$Given,\ x + y = \frac{dy}{dx} + 5 \qquad \Rightarrow \frac{dy}{dx} - y = x - 5$$

$$P = -1 \quad ; \quad Q = x - 5 \qquad \therefore \int Pdx = \int -1dx = -x$$

$$I.f. = e^{\int Pdx} = e^{-x} \qquad \therefore\ Sol.\ is\ ye^{-x} = \int (x-5)e^{-x}dx$$

$$= \int -x\,d\left(e^{-x}\right) - 5\int e^{-x}dx \quad = -xe^{-x} - \int -e^{-x}dx + 5e^{-x}$$

$$\therefore ye^{-x} = -xe^{-x} - e^{-x} - 5e^{-x} + c$$

$$given,\ y(0) = 2 \therefore 2 = 0 - 1 - 5.1 + c\ \therefore\ c = 8$$

$$\therefore y\,e^{-x} = -e^{x}(x+6) + 8$$

3. Find the equation of curve which passes through (0, 1) given that slope of that at any point = sum of abscissa and product of coordinates.

$$Given,\ \frac{dy}{dx} = x + xy \qquad \Rightarrow \frac{dy}{dx} - xy = x$$

$$P = -x \quad ; \quad Q = x \qquad \therefore \int Pdx = \int -x\,dx = \frac{-x^2}{2}$$

$$I.f. = e^{-x^2/2} \qquad \therefore\ Sol.\ is\ y.e^{-x^2/2} = \int e^{-x^2/2}dx$$

$$= \int e^{-x^2/2}\,d\left(e^{-x^2/2}\right) = -e^{-x^2/2} + c$$

$$given,\ y(0) = 1 \qquad \therefore 1, e^{0} = -e^{0} + c \quad \therefore\ c = 8$$

$$\therefore y\,e^{-x^2/2} = -e^{-x^2/2} + 2$$

4. Find the equation of the curve which passes through (0, 1) given that slope at any point on it $\left(\frac{dy}{dx}\right)$ satisfies (x – y) (dx + dy) = dx –dy.

$$Given,\ (x-y)(dx+dy)=dx-dy$$

$$\Rightarrow \int (dx+dy)=\int \frac{d(x-y)}{x-y} \quad \Rightarrow x+y=\log(x-y)+c$$

$$given,\ y(0)=-1 \quad \therefore 0-1=\log 1+c \quad \therefore c=-1 \quad \therefore x+y=\log(x-2)-1$$

5. Find the equation of curve which passes through (0, 2) given that product of ordinate and slope at that point = abscissa.

$$Given,\ \frac{ydy}{dx}=x$$

$$\Rightarrow \int y\ dy=\int x\,dx \quad \Rightarrow \frac{y^2}{2}=\frac{x^2}{2}+c$$

$$given,\ y(0)=-2 \quad \therefore \frac{(-2)^2}{2}=0+c \quad \therefore c=2$$

$$\therefore \frac{y^2}{2}=\frac{x^2}{2}+2 \quad \therefore x^2-y^2+4=0$$

6. Find the equation of a curve which passes through (1, 1).

$$Given,\ x\frac{dy}{dx}=(x+2)\ \ (y+2)$$

$$given, x\frac{dy}{dx}=(x+2)(y+2) \quad \therefore \int \frac{dy}{y+2}\int \left(\frac{x+2}{x}\right)dx$$

$$\Rightarrow \log(y+2)=\int \left(1+\frac{2}{x}\right)dx=x+2\log x+c$$

$$given,\ y(1)=1 \quad \therefore \log 3=1+2\log x+c$$

$$\therefore c=\log 3-1 \quad \therefore \log(y+2)=x+2\log x+\log 3-1$$

7. At any point P on a curve slope =2 (slope segment joining P & A (-4, -3). Find its equation if it passes through (-2, 1)

$$Given,\ \frac{dy}{dx}=2\left(\frac{x+3}{x+4}\right) \quad \Rightarrow \int \frac{dy}{y+3}=\int 2\frac{dx}{x+4}$$

$$\therefore \log(y+3)=2\log(x+4)+c$$

$$given, y(-2)=1 \quad \therefore \log y=2\log 2+c \quad \therefore c=0$$

$$\therefore \log(y+3)=2\log(x+4)$$

8. In a bank principal, increases continuously at the rate of 5% per year. In how many years of Rs.100 doubles itself? Use $\log_e^2=0.6931$

$$Given\ \frac{dp}{dt} = \frac{5}{100}.P \quad \therefore \int 20\frac{dp}{P} = \int dt \Rightarrow \log P = \frac{t}{20} + c$$

$$\therefore P = e^{t/20}.\,e^{c} \quad \therefore\ P = Ke^{t/20}$$

$$given\ P(0) = 1000 \qquad \therefore 1000 = ke^{0} \quad \therefore k = 1000$$

$$\therefore P = 1000e^{t/20} \quad \therefore 2000 = 1000e^{t/20} \quad \therefore\ 2 = e^{t/20} \quad \therefore \log 2 = \frac{t}{20}$$

$$\therefore\ t = 20\log 2\ =\ 20(0.6931) = 13.862$$

9. Find the equation of a curve whose differential equations is $y_1 = e^x \sin x$ given that it passes through origin.

$$y^1 = e^x \sin x \quad \Rightarrow \frac{dy}{dx} = e^x \sin x \quad \therefore \int dy = \int e^x \sin x\, dx$$

$$\therefore\ y = \int \frac{e^x}{2}(\sin x - \cos x) + c\ \ But\ y(0) = 0 \quad \therefore c = 1/2\ \therefore\ y = \frac{e^x}{2}(\sin x - \cos x) + \frac{1}{2}$$

10. Find equation of a curve, whose differential equation $\frac{dy}{dx} = (1 + x^2)(1 - y^2)$

given that it passes through A (0, ½)

$$given, \quad \frac{dy}{dx} = (1 + x^2)(1 - y^2) \quad \Rightarrow \int \frac{dy}{1 - y^2} = \int (1 + x^2)dx$$

$$\therefore\ \frac{1}{2}\log\left(\frac{1+y}{1-y}\right) = x + \frac{x^3}{3} + c$$

$$given\ \ y(0) = 1/2 \qquad \therefore \frac{1}{2}\log\left(\frac{1+1/2}{1-1/2}\right) = 0 + 0 + c$$

$$\therefore c = \frac{1}{2}\log 3 \quad \therefore \frac{1}{2}\log\left(\frac{1+y}{1-y}\right) = x + \frac{x^3}{3} + \frac{1}{2}\log 3$$