CHAPTER 8:

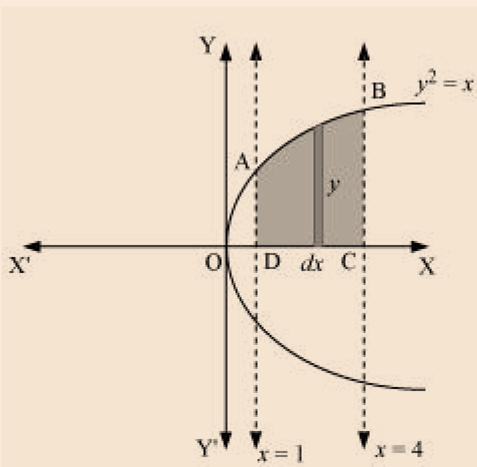
APPLICATION OF INTEGRALS

3 mark questions

Question 1:

Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis.

Answer :



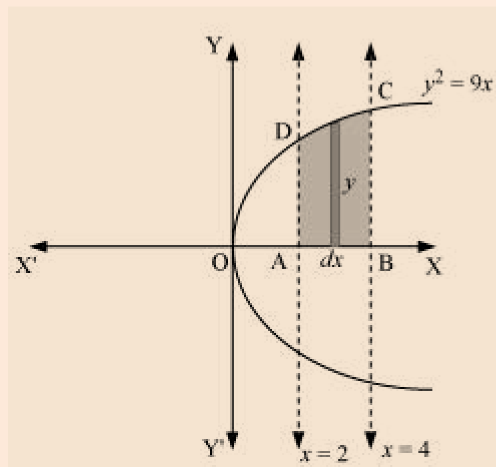
The area of the region bounded by the curve, $y^2 = x$, the lines, $x = 1$ and $x = 4$, and the x-axis is the area ABCD.

$$\text{Area of ABCD} = \int_1^4 y\,dx$$
$$= \int_1^4 \sqrt{x}\,dx$$
$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$
$$= \frac{2}{3}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]$$
$$= \frac{2}{3}[8-1]$$
$$= \frac{14}{3} \text{ units}$$

Question 2:

Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

Answer :



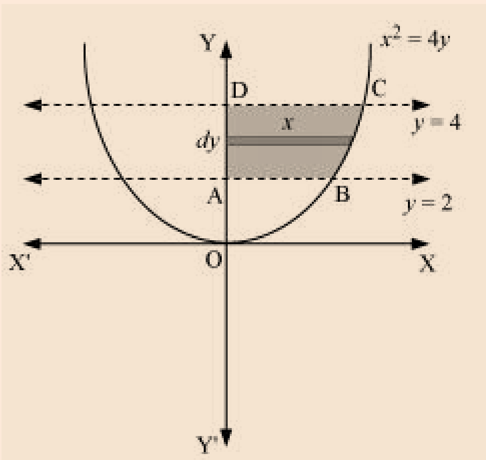
The area of the region bounded by the curve, $y^2 = 9x$, $x = 2$, and $x = 4$, and the x-axis is the area ABCD.

$$\text{Area of ABCD} = \int_2^4 y\,dx$$
$$= \int_2^4 3\sqrt{x}\,dx$$
$$= 3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$
$$= 2\left[x^{\frac{3}{2}}\right]_2^4$$
$$= 2\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$$
$$= 2\left[8 - 2\sqrt{2}\right]$$
$$= \left(16 - 4\sqrt{2}\right) \text{ units}$$

Question 3:

Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

Answer :



The area of the region bounded by the curve, $x^2 = 4y$, $y = 2$, and $y = 4$, and the y-axis is the area ABCD.

$$\text{Area of ABCD} = \int_2^4 x\,dy$$

$$= \int_2^4 2\sqrt{y}\,dy$$

$$= 2\int_2^4 \sqrt{y}\,dy$$

$$= 2\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

$$= \frac{4}{3}\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$$

$$= \frac{4}{3}\left[8 - 2\sqrt{2}\right]$$

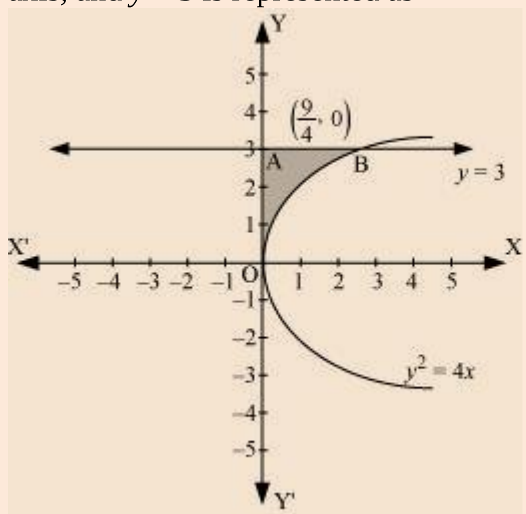
$$= \left(\frac{32 - 8\sqrt{2}}{3}\right) \text{ units}$$

Question 4:

Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

Answer :

The area bounded by the curve, $y^2 = 4x$, y-axis, and $y = 3$ is represented as



$$\therefore \text{Area OAB} = \int_0^3 x\,dy$$

$$= \int_0^3 \frac{y^2}{4}\,dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3$$

$$= \frac{1}{12}(27)$$

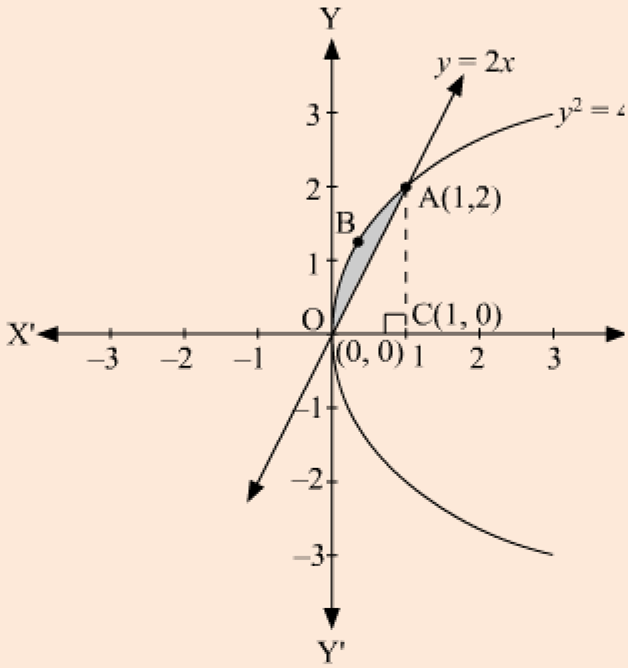
$$= \frac{9}{4} \text{ units}$$

Question 5:

Find the area lying between the curve $y^2 = 4x$ and $y = 2x$ is

Answer :

The area lying between the curve, $y^2 = 4x$ and $y = 2x$, is represented by the shaded area OBAO as



The points of intersection of these curves are O (0, 0) and A (1, 2).
We draw AC perpendicular to x-axis such that the coordinates of C are (1, 0).
∴ Area OBAO = Area (OCABO) – Area (ΔOCA)

$$= \int_0^1 2\sqrt{x}\,dx - \int_0^1 2x\,dx$$

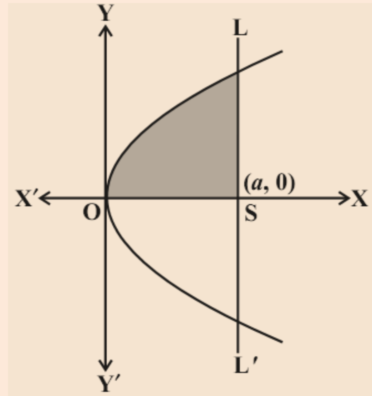
$$= 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 - 2\left[\frac{x^2}{2}\right]_0^1$$

$$= \left|\frac{4}{3} - 1\right|$$

$= \frac{1}{3}$ square units

Question 5.

Find area enclosed by the Parabola y^2=4ax and its latus rectum by integration
Solution: $y^2 = 4ax$ **----** (1) and
the equation of the Latus rectum is given by
x = a ……… (2)



From (2) and (1) $y^2 = 4a^2 \Rightarrow y = \pm 2a$

Required area A = 2 [area of OSP

$$= 2\int_0^a y.dx = 2\int_0^a 2\sqrt{a}\,.\,\sqrt{x}\,.\,dx$$

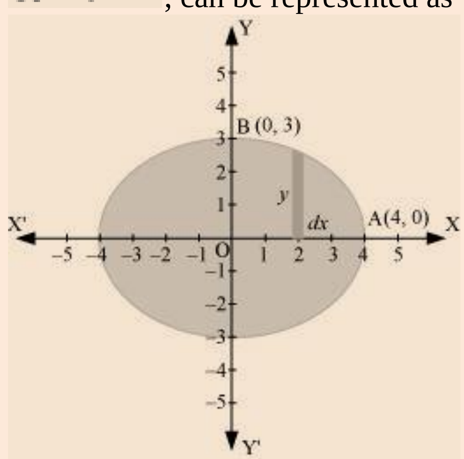
$$= 4\sqrt{a}\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^a = \frac{8\sqrt{a}}{3}\,[a\sqrt{a}] = \frac{8a^2}{3} \text{ Sq. units}$$

5 MARK QUESTIONS:

Question 1:

Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Answer :
The given equation of the ellipse, $\frac{x^2}{16} + \frac{y^2}{9} = 1$, can be represented as



It can be observed that the ellipse is symmetrical about x-axis and y-axis.

∴ Area bounded by ellipse = 4 × Area of OAB

$$\text{Area of OAB} = \int_0^4 y\,dx$$
$$= \int_0^4 3\sqrt{1-\frac{x^2}{16}}dx$$
$$= \frac{3}{4}\int_0^4 \sqrt{16-x^2}\,dx$$
$$= \frac{3}{4}\left[\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$
$$= \frac{3}{4}\left[2\sqrt{16-16}+8\sin^{-1}(1)-0-8\sin^{-1}(0)\right]$$
$$= \frac{3}{4}\left[\frac{8\pi}{2}\right]$$
$$= \frac{3}{4}[4\pi]$$
$$= 3\pi$$

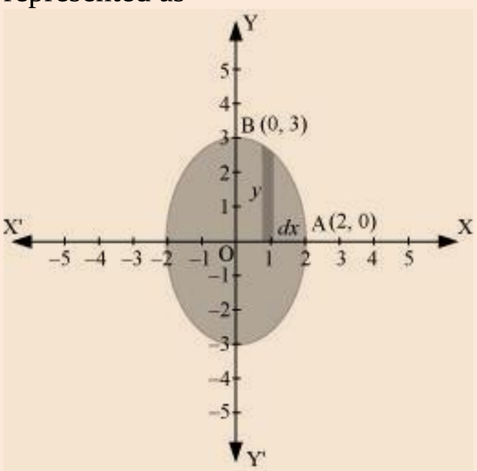
Therefore, area bounded by the ellipse = 4 × 3π = 12π units

Question 2:

Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

Answer :

The given equation of the ellipse can be represented as



$$\frac{x^2}{4}+\frac{y^2}{9}=1$$
$$\Rightarrow y = 3\sqrt{1-\frac{x^2}{4}} \quad ...(1)$$

It can be observed that the ellipse is symmetrical about *x*-axis and *y*-axis.

∴ Area bounded by ellipse = 4 × Area OAB

$$\therefore \text{Area of OAB} = \int_0^2 y\,dx$$
$$= \int_0^2 3\sqrt{1-\frac{x^2}{4}}dx \quad \text{[Using (1)]}$$
$$= \frac{3}{2}\int_0^2 \sqrt{4-x^2}\,dx$$
$$= \frac{3}{2}\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-}\frac{x}{2}\right]_0^2$$
$$= \frac{3}{2}\left[\frac{2\pi}{2}\right]$$
$$= \frac{3\pi}{2}$$

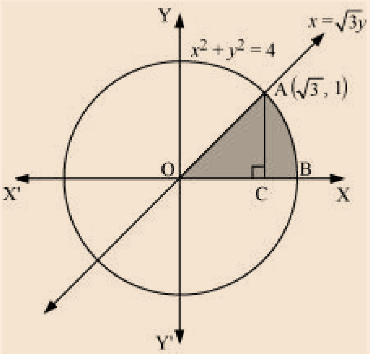
Therefore, area bounded by the ellipse = $4\times\frac{3\pi}{2} = 6\pi$ units

Question 3:

Find the area of the region in the first quadrant enclosed by *x*-axis, line $x=\sqrt{3}y$ and the circle $x^2+y^2=4$

Answer :

The area of the region bounded by the circle, $x^2+y^2=4, x=\sqrt{3}y$, and the *x*-axis is the area OAB.



The point of intersection of the line and the circle in the first quadrant is $\left(\sqrt{3},1\right)$.

Area OAB = Area ΔOCA + Area ACB

Area of OAC

$$=\frac{1}{2}\times OC\times AC=\frac{1}{2}\times\sqrt{3}\times 1=\frac{\sqrt{3}}{2} \qquad ...(1$$

Area of ABC $=\int_{\sqrt{3}}^{2} y\,dx$

$$=\int_{\sqrt{3}}^{2}\sqrt{4-x^2}\,dx$$

$$=\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$$

$$=\left[2\times\frac{\pi}{2}-\frac{\sqrt{3}}{2}\sqrt{4-3}-2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$=\left[\pi-\frac{\sqrt{3}\pi}{2}-2\left(\frac{}{3}\right)\right]$$

$$=\left[\pi-\frac{\sqrt{3}}{2}-\frac{2\pi}{3}\right]$$

$$=\left[\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right] \qquad ...(2)$$

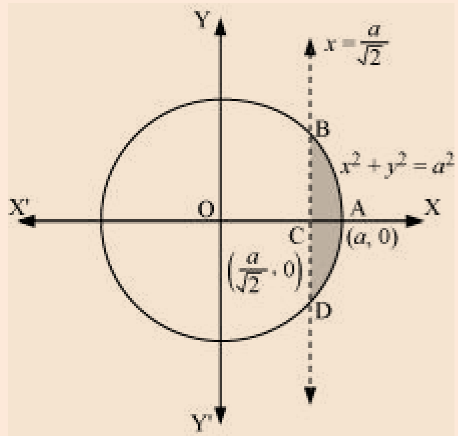
Therefore, area enclosed by *x*-axis, the line $x=\sqrt{3}y$, and the circle $x^2+y^2=4$ in the first quadrant = $\frac{\sqrt{3}\pi}{2}+\frac{}{3}-\frac{3\sqrt{\ }\pi}{2}=\frac{\pi}{3}$ units

Question 7:

Find the area of the smaller part of the circle $x^2+y^2=a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$

Answer :

The area of the smaller part of the circle, $x^2+y^2=a^2$, cut off by the line, $x=\frac{a}{\sqrt{2}}$, is the area ABCDA.



It can be observed that the area ABCD is symmetrical about *x*-axis.

∴ Area ABCD = 2 × Area ABC

$$\text{Area of } ABC=\int_{\frac{a}{\sqrt{2}}}^{a} y\,dx$$

$$=\int_{\frac{a}{\sqrt{2}}}^{a}\sqrt{a^2-x^2}\,dx$$

$$=\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a}$$

$$=\left[\frac{a^2}{2}\left(\frac{\pi}{2}\right)-\frac{a}{2\sqrt{2}}\sqrt{a^2-\frac{a^2}{2}}-\frac{a^2}{2}\sin^{-1}\left(\frac{}{\sqrt{2}}\right)\right]$$

$$=\frac{a^2\pi}{4}-\frac{a}{2\sqrt{2}}\cdot\frac{a}{\sqrt{2}}-\frac{a^2}{2}\left(\frac{\pi}{4}\right)$$

$$=\frac{a^2\pi}{4}-\frac{a^2}{4}-\frac{a^2\pi}{8}$$

$$=\frac{a^2}{4}\left[\pi-1-\frac{\pi}{2}\right]$$

$$=\frac{a^2}{4}\left[\frac{\pi}{2}-1\right]$$

$$\Rightarrow \text{Area } ABCD=2\left[\frac{a^2}{4}\left(\frac{\pi}{2}-1\right)\right]=\frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$$

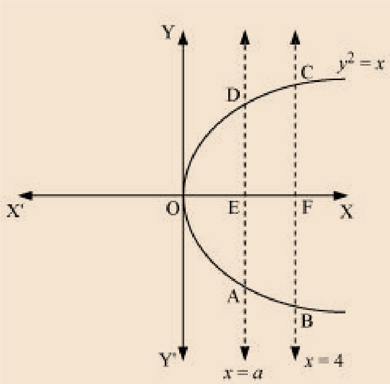
Therefore, the area of smaller part of the circle, $x^2 + y^2 = a^2$, cut off by the line, $x = \frac{a}{\sqrt{2}}$, is $\frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$ units.

Question 8:

The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

Answer :

The line, $x = a$, divides the area bounded by the parabola and $x = 4$ into two equal parts.
∴ Area OAD = Area ABCD



It can be observed that the given area is symmetrical about x-axis.
⇒ Area OED = Area EFCD

$$\text{Area OED} = \int_0^a y\,dx$$

$$= \int_0^a \sqrt{x}\,dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$

$$= \frac{2}{3}(a)^{\frac{3}{2}} \quad \ldots(1)$$

$$\text{Area of EFCD} = \int_a^4 \sqrt{x}\,dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_a^4$$

$$= \frac{2}{3}\left[8 - a^{\frac{3}{2}}\right] \quad \ldots(2)$$

From (1) and (2), we obtain

$$\frac{2}{3}(a)^{\frac{3}{2}} = \frac{2}{3}\left[8 - (a)^{\frac{3}{2}}\right]$$

$$\Rightarrow 2 \cdot (a)^{\frac{3}{2}} = 8$$

$$\Rightarrow (a)^{\frac{3}{2}} = 4$$

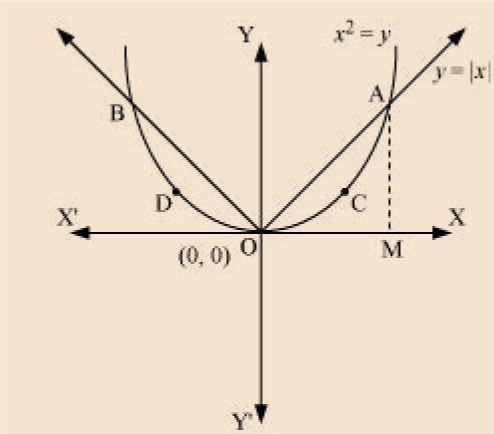
$$\Rightarrow a = (4)^{\frac{2}{3}}$$

Therefore, the value of a is $(4)^{\frac{2}{3}}$.

Question 9:

Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$

Answer :
The area bounded by the parabola, $x^2 = y$,and the line, $y = |x|$, can be represented as



The given area is symmetrical about y-axis.
∴ Area OACO = Area ODBO
The point of intersection of parabola, $x^2 = y$, and line, $y = x$, is A (1, 1).
Area of OACO = Area ΔOAM – Area OMACO
Area of ΔOAM

$= \frac{1}{2} \times OM \times AM$ $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

Area of OMACO

$= \int_0^1 y\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

⇒ Area of OACO = Area of ΔOAM – Area of OMACO

$= \frac{1}{2} - \frac{1}{3}$

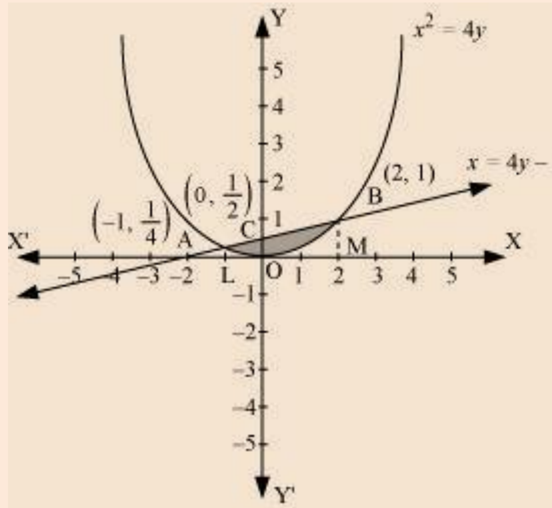
$= \frac{1}{6}$

Therefore, required area = $2\left[\frac{1}{6}\right] = \frac{1}{3}$ units

Question 10:

Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$

Answer :
The area bounded by the curve, $x^2 = 4y$, and line, $x = 4y - 2$, is represented by the shaded area OBAO.



Let A and B be the points of intersection of the line and parabola.

Coordinates of point A are $\left(-1, \frac{1}{4}\right)$.
Coordinates of point B are (2, 1).
We draw AL and BM perpendicular to x-axis.
It can be observed that,
Area OBAO = Area OBCO + Area OACO … (1)
Then, Area OBCO = Area OMBC – Area OMBO

$= \int_0^2 \frac{x+2}{4}\,dx - \int_0^2 \frac{x^2}{4}\,dx$

$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$

$= \frac{1}{4}[2+4] - \frac{1}{4}\left[\frac{8}{3}\right]$

$= \frac{3}{2} - \frac{2}{3}$

$= \frac{5}{6}$

Similarly, Area OACO
= Area OLAC – Area OLAO

$$= \int_{-1}^{0} \frac{x+2}{4}dx - \int_{-1}^{0} \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{0} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{0}$$

$$= -\frac{1}{4}\left[\frac{(-1)^2}{2} + 2(-1)\right] - \left[-\frac{1}{4}\left(\frac{(-1)^3}{3}\right)\right]$$

$$= -\frac{1}{4}\left[\frac{1}{2} - 2\right] - \frac{1}{12}$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{12}$$

$$= \frac{7}{24}$$

Therefore, required area =

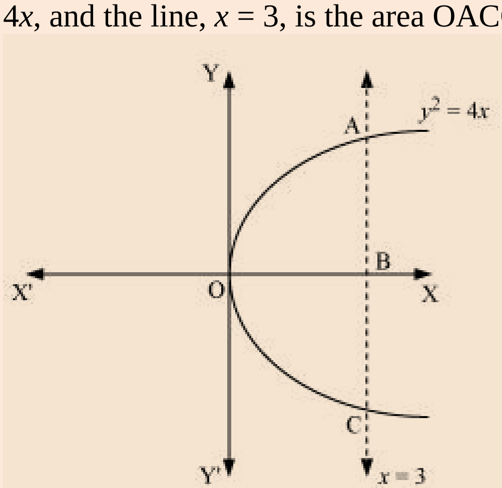
$$\left(\frac{5}{6} + \frac{7}{24}\right) = \frac{9}{8} \text{ units}$$

Question 11:

Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$

Answer :

The region bounded by the parabola, $y^2 = 4x$, and the line, $x = 3$, is the area OACO.



The area OACO is symmetrical about x-axis.

∴ Area of OACO = 2 (Area of OAB)

$$\text{Area OACO} = 2\left[\int_0^3 y\,dx\right]$$

$$= 2\int_0^3 2\sqrt{x}\,dx$$

$$= 4\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$$

$$= \frac{8}{3}\left[(3)^{\frac{3}{2}}\right]$$

$$= 8\sqrt{3}$$

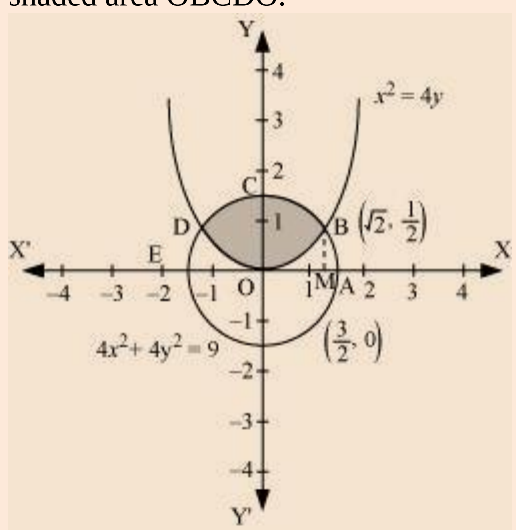
Therefore, the required area is $8\sqrt{3}$ units.

Question 12:

Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$

Answer :

The required area is represented by the shaded area OBCDO.



Solving the given equation of circle, $4x^2 + 4y^2 = 9$, and parabola, $x^2 = 4y$, we obtain the point of intersection as

$$B\left(\sqrt{2}, \frac{1}{2}\right) \text{ and } D\left(-\sqrt{2}, \frac{1}{2}\right).$$

It can be observed that the required area is symmetrical about y-axis.

$\therefore$ Area OBCDO = 2 × Area OBCO
We draw BM perpendicular to OA.

Therefore, the coordinates of M are $\left(\sqrt{2}, 0\right)$.
Therefore, Area OBCO = Area OMBCO – Area OMBO
Therefore, the required area OBCDO is

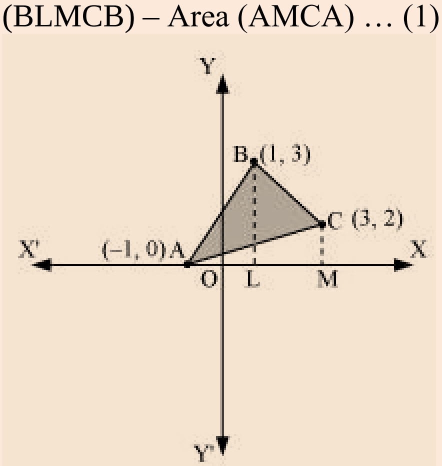
$$\left(2\times\frac{1}{2}\left[\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right]\right)=\left[\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right.$$

units

Question :13

Using integration finds the area of the region bounded by the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

Answer :
BL and CM are drawn perpendicular to *x*-axis.
It can be observed in the following figure that,
Area (ΔACB) = Area (ALBA) + Area (BLMCB) – Area (AMCA) … (1)



Equation of line segment AB is

$$y-0=\frac{3-0}{1+1}(x+1)$$

$$y=\frac{3}{2}(x+1)$$

$$\therefore \text{Area}(\text{ALBA})=\int_{-1}^{1}\frac{3}{2}(x+1)\,dx=\frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1}=$$

Equation of line segment BC is

$$y-3=\frac{2-3}{3-1}(x-1)$$

$$y=\frac{1}{2}(-x+7)$$

$$\therefore \text{Area}(\text{BLMCB})=\int_{1}^{3}\frac{1}{2}(-x+7)\,dx=\frac{1}{2}\left[-\frac{x^2}{2}+7x\right]_{1}^{3}=\frac{1}{2}\left[-\right.$$

Equation of line segment AC is

$$y-0=\frac{2-0}{3+1}(x+1)$$

$$y=\frac{1}{2}(x+1)$$

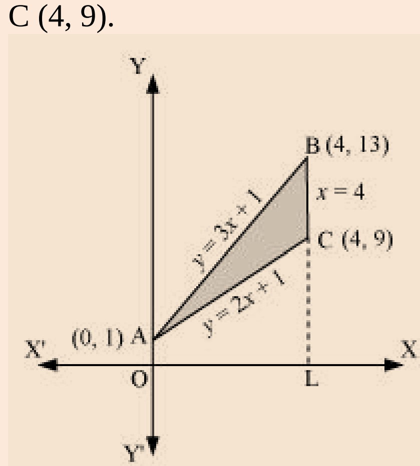
$$\therefore \text{Area}(\text{AMCA})=\frac{1}{2}\int_{-1}^{3}(x+1)\,dx=\frac{1}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{3}=\frac{1}{2}\left[\ldots+3-\right.$$

Therefore, from equation (1), we obtain
Area (ΔABC) = (3 + 5 – 4) = 4 units

Question 14:

Using integration find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Answer :
The equations of sides of the triangle are $y = 2x + 1$, $y = 3x + 1$, and $x = 4$.
On solving these equations, we obtain the vertices of triangle as A(0, 1), B(4, 13), and C (4, 9).



It can be observed that,

Area (ΔACB) = Area (OLBAO) –Area (OLCAO)

$$= \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx$$

$$= \left[\frac{3x^2}{2}+x\right]_0^4 - \left[\frac{2x^2}{2}+x\right]_0^4$$

$$= (24+4)-(16+4)$$

$$= 28-20$$

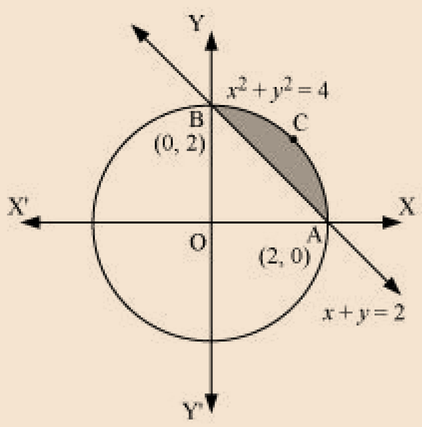
$$= 8 \text{ units}$$

Question 15:

Find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

Answer :

The smaller area enclosed by the circle, $x^2 + y^2 = 4$, and the line, $x + y = 2$, is represented by the shaded area ACBA as



It can be observed that,
Area ACBA = Area OACBO – Area (ΔOAB)

$$= \int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (2-x)\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x-\frac{x^2}{2}\right]_0^2$$

$$= \left[2\cdot\frac{\pi}{2}\right]-[4-2]$$

$$= (\pi-2) \text{ units}$$