Definite Integrals

One marks questions :

Evaluate the following integrals :-

(1). $\int_{-\pi/2}^{\pi/2} \sin^5 x \ \ dx$

Soln: $\int_{-\pi/2}^{\pi/2} \sin^5 x \ \ dx = 0 \qquad \because \ \sin^5 x \quad \textit{is an odd function.}$

(2). $\int_0^1 \frac{2x}{1+x^2} \ \ dx.$

Soln: $\int_0^1 \frac{2x}{1+x^2} \ \ dx \ = [\log(1+x^2)]_0^1 \ = \ \log 2$

(3). $\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}} \ \ dx \ .$

Soln: $\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}} \ \ dx \ = \ (\sin^{-1} x)_0^{1/\sqrt{2}} \ = \ \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1} 0 \ = \ \frac{\pi}{4}.$

(4). $\int_0^{\pi/3} (3\sin x - 4\sin^3 x) \ \ dx.$

Soln: $\int_0^{\pi/3} (3\sin x - 4\sin^3 x) \ \ dx \ = \ \int_0^{\pi/3} \sin 3x \ \ dx \ = -\frac{(\cos 3x)_0^{\pi/3}}{3} \ = \ \frac{-1}{3}\,[-1-1] \ = \ \frac{2}{3}$

(5). $\int_{-1}^{1} \frac{x^3}{1+x^2} \ \ dx.$

Soln: $\int_{-1}^{1} \frac{x^3}{1+x^2} \ \ dx \ = \ 0$ because $\frac{x^3}{1+x^2}$ $\textit{is an odd function.}$

Two marks questions :

(1) Evaluate $\int_{-5}^{5} |x+2|\, dx$

Soln: Let $x+2 = t \ \Rightarrow dx = dt \ when \ x = 5 \Rightarrow t = 7; when \ x = -5 \Rightarrow t = -3$

$\int_{-5}^{5} |x+2|\, dx \Rightarrow \int_{-3}^{7} |t|\, dt \Rightarrow \frac{1}{2}(t|t|)_{-3}^{7} \Rightarrow \frac{1}{2}(7|7| - (-3)|-3|) \Rightarrow 29$

(2) Evaluate $\int_2^3 \frac{x}{x^2+1}\, dx$

Soln: Let $x^2 + 1 = t \Rightarrow 2x\, dx = dt \Rightarrow xdx = \frac{dt}{2} \ when \ x = 2 \Rightarrow t = 5; when \ x = 3 \Rightarrow t = 10$

$\int_2^3 \frac{x}{x^2+1}\, dx = \int_5^{10} \frac{1}{t}\frac{dt}{2} = \frac{1}{2}(\log t)_5^{10} = \frac{1}{2}\,(\log 10 - \log 5) = \frac{1}{2}\log 2.$

(3) Evaluate $\int_0^{\frac{\pi}{2}} \cos 2x\, dx.$

Soln : $\int_0^{\frac{\pi}{2}} \cos 2x\, dx = \frac{1}{2}(\sin 2x)_0^{\frac{\pi}{2}} = \frac{1}{2}(0-0) = 0$

(4) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$

Soln: Let $\cos x = t \Rightarrow \sin x \ \ dx = -dt; when \ x = 0 \Rightarrow t = 1; when \ x = \frac{\pi}{2} \Rightarrow t = 0$

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2} = -(\tan^{-1} t)_1^0 = -\left(0 - \frac{\pi}{4}\right) = \frac{\pi}{4}$

(5) $\int_0^{\pi} \ x \ \ \sin x \ \ \cos^2 x \ \ dx.$

Soln: $\textit{Let} \ \ I = \int_0^{\pi} \ x \ \ \sin x \ \ \cos^2 x \ \ dx \ = \int_0^{\pi} (\pi - x) \ \sin x \ \cos^2 x \ \ dx.$

$\therefore \ \ 2\, I = \int_0^{\pi} (x + \pi - x) \ \ \sin x \ \cos^2 x \ \ dx \ = \ \pi \ \int_0^{\pi} \cos^2 x. \ \sin x \ \ dx$

$= \ \pi \ \int_1^{-1} \ t^2 \ \ (-dt)$ here $\cos x \ = \ t \Rightarrow -\sin x\, dx = dt$ when $x = 0 \Rightarrow t = 1; x = \pi \Rightarrow t = -1$

$$2I = \pi \int_{-1}^{1} t^2\,dt = \pi.\frac{(t^3)_{-1}^{1}}{3} = \frac{\pi}{3}\,(1+1) = \frac{2\pi}{3} \Rightarrow I = \frac{\pi}{3}.$$

Six marks questions :

(1) Prove that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ and evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$.

Soln: Consider $\int_a^b f(a+b-x)\,dx$

$put\ a+b-x=t$

$\mapsto -dx = dt.\ \ At\ x=a.,t=a+b-a=b\ and\ at\ x=b.,\ t=a+b-b=a$

$\therefore RHS = \int_a^b f(a+b-x)\,dx = \int_b^a f(t)\,(-dt)$ $\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$

$= \int_a^b f(t)\,dt = \int_a^b f(x)\,dx = LHS.$ [Definite integrals are independent of variable]

Consider $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$

Let I $= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}+\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}\,dx$ ....(1)

$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$...................(2)

Adding (1) and (2)

$2\,I = I + I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$

$2I = (x)_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$ $\therefore$ $I = \frac{\pi}{12}$

(2) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ and hence evaluate $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$.

Soln : Consider $\int_0^a f(a-x)\,dx$

Let $a-x=t \Rightarrow dx=-dt; when\ x=0 \Rightarrow t=a; when\ x=a \Rightarrow t=0$

$\int_0^a f(a-x)\,dx = \int_a^0 f(t)(-dt)$ $\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$

$= \int_0^a f(t)\,dt = \int_0^a f(x)\,dx$ [Definite integrals are independent of variable]

Let I $= \int_0^{\pi/4} \log(1+\tan x)\,dx = \int_0^{\pi/4} \log\left(1+\tan\langle\frac{\pi}{4}-x\rangle\right)\,dx = \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)\,dx$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)\,dx$

$$\therefore 2I = I + I = \int_0^{\pi/4} \log(1+\tan x)\,dx + \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)\,dx = \int_0^{\pi/4} \log 2\,dx = \log 2\,(x)_0^{\pi/4} = \frac{\pi}{4}\log 2$$

$\therefore$ I $= \frac{\pi}{8}\log 2.$

(3) Prove that $\int_0^{2a} f(x)\,dx = \begin{cases} 2.\int_0^a f(x)\,dx, & if\ f(2a-x)=f(x), \\ 0 & , if\ f(2a-x) = -f(x) \end{cases}$ and hence evaluate $\int_0^{\pi} \frac{1}{a^2\cos^2 x+b^2\sin^2 x}\,dx$.

Soln : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \ldots..(1)$

$Put\ 2a-x=t \mapsto dx=-dt$

Consider $\int_a^{2a} f(x)\,dx.$ $At\ x=a.,t=a, and\ at\ x=2a,t=0$

$= \int_a^0 f(2a-t)(-dt)$

$= \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx \ldots.(2)$

Substituting (2) in (1)., we get : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

If $f(2a - x) = f(x)$ the above result $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx.,$

And if $f(2a - x) = -f(x).,$ it reduces to: $\int_0^{2a} f(x)\,dx = 0.$

Let I $= \int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\,dx = 2\int_0^{\pi/2} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\,dx$

$= 2\int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\,dx$

Put b tan $x = t,\ \Rightarrow\ b\sec^2 x\,dx = dt, \therefore\ \sec^2 x\,dx = \frac{dt}{b},\ At\ x = 0.,\ t = 0, At\ x = \frac{\pi}{2},\ t \to \infty.$

$I = 2\int_0^{\infty} \frac{dt}{b(a^2+t^2)} = 2\,\frac{1}{b}\frac{1}{a}\left(\tan^{-1}\frac{t}{a}\right)_0^{\infty} = \frac{2}{ab}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{ab}$

(4) Prove that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ and hence evaluate $\int_{-1}^{2} |x^3 - x|dx$

Soln: *et* $F(x)$ *be the antiderivative of* $f(x)$.

$$Then.,\ RHS = \int_a^c f(x)\,dx + \int_c^b f(x)dx = F(c) - F(a) + F(b) - F(c)$$

$$\int_a^b f(x)\,dx = F(b) = F(a) = \int_a^b f(x)\,dx = LHS.$$

Clearly, $x^3 - x\,x(x-1)(x+1)$ will be positive between -1 & 0, negative between 0 & 1 and again positive between 1 & 2.

$$\therefore \int_{-1}^{2} (x^3 - x)\,dx = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx + \int_{1}^{2} (x^3 - x)\,dx.$$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4}\right)_{0}^{1} + \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{1}^{2}$$

$$= 0 - \left[\frac{1}{4} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{4}\right] + \left\{(4-2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right\}$$

$$= +\frac{1}{4} + \frac{1}{4} + 2 + \frac{1}{4} = 2\,\frac{3}{4} = \frac{11}{4}$$

(5) Prove that $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx., & \text{if f is an even function} \\ 0 & , if\ f\ is\ an\ odd\ function \end{cases}$ and hence evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\,dx$.

Soln: consider $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \ldots .(1)$

$Put\ x = -t \Rightarrow x = -dt$

Consider $\int_{-a}^{0} f(x)\,dx \quad At\ x = -a.,t = a\ And, at\ x = 0.,\ t = 0.$

$$= \int_a^0 f(-t)(-dt) = \int_0^a f(-t)\,dt = \int_0^a f(-x)\,dx = \begin{cases} \int_0^a f(x)\,dx & \text{if f is an even function, and,} \\ -\int_0^a f(x)dx & if\ f\ is\ an\ odd\ function. \end{cases} \ldots .(2)$$

Substitutins eq.(2) in eq. (1)., we get :

$$\int_{-a}^{a} f(x)\,dx = \begin{pmatrix} \int_0^a f(x)\,dx, & if\ f\ is\ even \\ -\int_0^a f(x)\,dx, & if\ f\ is\ odd \end{pmatrix} + \int_0^a f(x)\,dx$$

$$= \begin{cases} 2.\int_0^a f(x)\,dx, & \text{if f is an even function, and,} \\ 0 & , if\ f\ is\ an\ odd\ function. \end{cases}$$

Hence the value of $\int_{\frac{-\pi}{2}}^{\pi/2} \sin^7 x\,dx = 0 \qquad [\therefore\ \sin^7(-x) = -\sin^7 x \Rightarrow it\ is\ an\ odd\ function]$

(6) Prove that $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$ and hence evaluate $\int_0^{\pi/2} \log\sin x\, dx$.

Soln : Consider $\int_0^a f(a-x)\ dx$

Let $a-x=t \Rightarrow dx=-dt; when\ x=0 \Rightarrow t=a; when\ x=a \Rightarrow t=0$

$\int_0^a f(a-x)\ dx = \int_a^0 f(t)(-dt)$ $\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\ dx\ \right]$

$= \int_0^a f(t)\,dt = \int_0^a f(x)\,dx$ [Definite integrals are independent of variable]

$$Let\ I = \int_0^{\pi/2} \log\sin x\ \ dx = \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2}-x\right)\ dx = \int_0^{\pi/2} \log\cos x\ \ dx$$

$2\,\mathrm{I} = \int_0^{\pi/2} \log\sin x\ \ dx + \int_0^{\pi/2} \log\cos x\ \ dx\ = \int_0^{\pi/2} \log(\sin x\ \cos x)\ \ dx$

$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)\ \ dx = \int_0^{\pi/2} \log\sin 2x\ \ dx\ -\ \int_0^{\pi/2} \log 2\,dx \ldots.(1)$

Consider $\int_0^{\pi/2} \log\sin 2x\ \ dx$ $\qquad Put\ \ 2x\ =\ t\ \mapsto\ dx\ =\ \frac{dt}{2}$

$At\ \ x\ =\ 0, t\ =\ 0. And,$

$= \int_0^{\pi} \log\sin t\ \frac{dt}{2}$ $\qquad At\ x = {}^{\pi}/_{2},\ \ t\ =\ \pi.$

$$= \int_0^{\pi/2} \log\sin t\ dt \qquad \left[\therefore \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx, \qquad if\ f(2a-x)=f(x)\right]$$

$$= \int_0^{\pi/2} \log\sin 2x\ \ \ dx =\ \ I \ldots\ldots(2)$$

$Substitutins\ eq\ (2)\ \ in\ eq.(1).,\ we\ get$: $2\,\mathrm{I} =\ \mathrm{I} - \int_0^{\pi/2} \log 2\ \ \ dx\ \therefore\ \mathrm{I} =\ -\log 2\ \ (x)_0^{\pi/2}\ =\ -\frac{\pi}{2}\log 2$

(7) Prove that $\int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$ when $f(2a-x)=f(x)$ and hence evaluate $\int_0^{\pi} |\cos x|\,dx$.

Soln : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \ldots..(1)$

$Put\ 2a-x=t \mapsto dx=-dt$

Consider $\int_a^{2a} f(x)\ \ dx.$ $\qquad At\ \ x=a.,t=a, and\ at\ x\ =\ 2a, t\ =\ 0$

$= \int_a^0 f(2a-t)(-dt)$

$= \int_0^a f(2a-t)\ dt\ =\ \int_0^a f(2a-x)\ dx \ldots.(2)$

$Substituting\ \ (2)\ \ in\ (1).,\ we\ get$: $\int_0^{2a} f(x)\ dx = \int_0^a f(x)\ dx\ +\ \int_0^a f(2a-x)\ \ dx$

$If\ f(2a\ -\ x)\ =\ f(x)$ the above result becomes $\qquad \int_0^{2a} f(x)\ \ dx = 2\int_0^a f(x)\ dx$

Consider $I = \int_0^{\pi} |\cos x|\,dx = 2\int_0^{\frac{\pi}{2}} |\cos x|\,dx$ $\qquad$ because $|\cos(\pi-x)| = |-\cos x| = |\cos x|$

$\therefore I = 2\int_0^{\frac{\pi}{2}} \cos x\,dx$ $\qquad$ because $\cos x$ is positive in 1st quadrant.

$I = 2\ (\sin x)_0^{\frac{\pi}{2}} = 2(1-0) = 2.$

(8) Prove that $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\ \ \mathrm{dx.},\ \ \text{if f is an even function} \\ 0\ \ \ \ \ \ \ \ ,\ if\ f\ is\ an\ odd\ function \end{cases}$ and hence evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\,dx$

Soln: consider $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\ \ dx + \int_0^a f(x)\,dx \ldots.(1)$

$Put\ x\ =\ -t \mapsto x\ =\ -dt$

Consider $\int_{-a}^{0} f(x)\,dx$ $\qquad At\ x\ =\ -a.,t\ =\ a\ \ And, at\ \ x\ =\ 0.,\ \ t\ =\ 0.$

$$= \int_a^0 f(-t)(-dt) = \int_0^a f(-t)\ \ dt\ =\ \int_0^a f(-x)\ dx = \begin{cases} \int_0^a f(x)\,\mathrm{dx}\quad \text{if f is an even function, and,} \\ -\int_0^a f(x)dx\quad if\ \ f\ \ is\ an\ odd\ function. \\ \qquad\qquad\qquad \ldots.(2) \end{cases}$$

$$Substitutins\ eq.(2)\ in\ eq.\ (1).,\ we\ get:$$

$$\int_{-a}^{a} f(x)\,dx = \begin{pmatrix} \int_0^a f(x)\,dx, & if\ f\ is\ even \\ -\int_0^a f(x)\,dx, & if\ f\ is\ odd \end{pmatrix} + \int_0^a f(x)\,dx$$

$$= \begin{cases} 2.\int_0^a f(x)\,\mathrm{dx}, & \text{if f is an even function, and,} \\ 0 & ,\ if\ f\ is\ an\ odd\ function. \end{cases}$$

Consider $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\,dx$ since $x^3, x\cos x, \tan^5 x$ odd functions

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\ dx = (x)_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} - \left(\frac{\pi}{2}\right) = \pi.$

(9) Prove that $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$ and hence evaluate $\int_0^{\pi/2} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$.

Soln: Consider $\int_0^a f(a-x)\ dx$

Let $a - x = t \Rightarrow dx = -dt; when\ x = 0 \Rightarrow t = a; when\ x = a \Rightarrow t = 0$

$\int_0^a f(a-x)\ dx = \int_a^0 f(t)(-dt)$ $\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\ dx\ \right]$

$= \int_0^a f(t)\,dt = \int_0^a f(x)\,dx$ [Definite integrals are independent of variable]

Consider $I = \int_0^{\pi/2} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$ ....(1)

$= \int_0^{\pi/2} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) + \cos^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)} dx$

$I = \int_0^{\pi/2} \frac{\cos^{\frac{3}{2}} x}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x} dx$ ....(2)

Adding (1) and (2)

$2I = \int_0^{\pi/2} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} + \frac{\cos^{\frac{3}{2}} x}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x}\ dx = \int_0^{\pi/2} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}\ dx = \int_0^{\pi/2} dx = (x)_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

Hence $I = \frac{\pi}{4}$.

(10) Prove that $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$ and hence evaluate $\int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\,dx$.

Soln: Consider $\int_0^a f(a-x)\ dx$

Let $a - x = t \Rightarrow dx = -dt; when\ x = 0 \Rightarrow t = a; when\ x = a \Rightarrow t = 0$

$\int_0^a f(a-x)\ dx = \int_a^0 f(t)(-dt)$ $\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\ dx\ \right]$

$= \int_0^a f(t)\,dt = \int_0^a f(x)\,dx$ [Definite integrals are independent of variable]

Consider $I = \int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\,dx = \int_0^{\pi/2} (\log\sin^2 x - \log\sin 2x)\,dx$

$= \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{\sin 2x}\right)\ dx = \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{2\sin x\cos x}\right) dx = \int_0^{\pi/2} \log\left(\frac{1}{2}\tan x\right) dx$ ....(1)

$I = \int_0^{\pi/2} \log\left(\frac{1}{2}\left(\tan\left(\frac{\pi}{2} - x\right)\right)\right) dx = \int_0^{\pi/2} \log\left(\frac{1}{2}\cot x\right)$..............(2)

Adding (1) and (2)

$$2I = \int_0^{\pi/2} \log\left(\frac{1}{2}\tan x\right) + \log\left(\frac{1}{2}\cot x\right) dx = \int_0^{\pi/2} \log\left(\frac{1}{2}\tan x \cdot \frac{1}{2}\cot x\right) dx = \int_0^{\pi/2} \log\left(\frac{1}{4}\right) dx = \log\frac{1}{4}(x)_0^{\pi/2}$$

$2I = \frac{\pi}{2}\log\frac{1}{4} \Rightarrow I = \frac{\pi}{4}\log\frac{1}{4}.$