APPLICATION OF DERIVATIVES

TWO MARK QUESTIONS:

1) Find the rate of change of the area of a circle w.r.t to its radius ‘r’ when r = 4 cm?

Ans: Area of circle $A = \pi r^2$, $dA/dr = ?$ when r = 4 cm

Differentiate w.r.t. ‘r’

$$dA/dr = \pi(2r)$$
$$= \pi(2)(4)$$
$$= 8\pi \text{ sq. cms}$$

Therefore area of the circle is increasing at the rate of 8π sq. cms.

2) An edge of a variable cube is increasing at the rate of 3cm/s. How fast is the volume of the cube increasing when the edge is 10cm long?

Ans: Volume of a cube $V = x^3$., Given: $dx/dt = 3$cm/s. $dV/dt = ?$ when x = 10cm

Didifferentiate w.r.t ‘t’

$$dV/dt = 3x^2(dx/dt)$$
$$= 3(10)^2 . (3)$$
$$= 900 \text{ c.c/s}$$

Therefore volume of the cube increasing at the rate of 900 c.c/s.

3) Show that the function $f(x) = x^3 - 3x^2 + 4x$, $x \in R$ is strictly increasing on R.

Ans: $f(x) = x^3 - 3x^2 + 4x$

Differentiate w.r. t x

$$f'(x) = 3x^2 - 6x + 4$$
$$= 3(x^2 - 2x + 1) + 1$$
$$= 3(x-1)^2 + 1 > 0, \forall x \in R$$

Therefore f is strictly increasing on R.

4) Show that the function $f(x) = e^{2x}$ is strictly increasing on R.

Ans: $f(x) = e^{2x}$,

Differentiate w.r.t x

$$f'(x) = 2. e^{2x}$$

clearly $f'(x) > 0 \ \forall \ x \in R$ (since exponential function is always positive)

Therefore f is strictly increasing on R.

5) Find the intervals in which $f(x) = x^2 + 2x - 5$ is strictly increasing or decreasing.

Ans: $f(x) = x^2 + 2x - 5$

Differentiate w.r.t x

$f'(x) = 2x + 2$

$= 2(x+1)$

Now $f'(x) = 0, \Rightarrow 2(x+1) = 0$

$\therefore x = -1.$

Now x = -1 divides the real line into 2 disjoint intervals namely $(-\infty, -1)$ and $(-1, \infty)$.

In $(-\infty, -1)$, $f'(x) < 0$

In $(-1,\infty)$, $f'(x) > 0$.

$\therefore$ f is strictly decreasing in $(-\infty, -1)$ and f is strictly increasing in $(-1,\infty)$.

6) Find the slope of the tangent to the curve $y = (x-1)/(x-2)$, $x \neq 2$ at $x = 10$.

Ans: $y = \frac{x-1}{x-2}$

Differentiate w.r.t x

$dy/dx = (x-1)\,[(-1)/(x-2)^2] + [1/(x-2)](1)$

slope of tangent = $dy/dx \mid x = 10$

$= (10-1)[(-1)/(10-2)^2] + [1/(10-2)](1)$

$= -9/64 + 1/8 \quad = \quad -1/64$

7) Find the points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to x axis.

Ans: $y = x^3 - 3x^2 - 9x + 7$.

Differentiate w. r. t. x

$dy/dx = 3x^2 - 6x - 9$ = slope of the tangent.

Given tangent is parallel to x axis.

Slope of the tangent = slope of x axis.

$3x^2 - 6x - 9 = 0$

$X^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\Rightarrow x = 3, x = -1$

When $x = 3$, $y = (3)^3 - 3(3)^2 - 9(3) + 7 = -20$

When $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 7 = 12$

Therefore the points are (3,-20) , (-1,12) .

8) Using differentials, find approximate value of $\sqrt{25.3}$ up to 3 decimal places.

Ans: $y = \sqrt{x}$, Let $x = 25$ and $\Delta x = 0.3$

Then $\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$

$= \sqrt{25.3} - \sqrt{25}$

$\sqrt{25.3} = \Delta y + 5$

Now $dy = (dy/dx)\ \Delta x = (1/2\sqrt{x})\ (0.3)$

$= (1/2\sqrt{25})\ (0.3) = 0.3/10 = 0.03.$

Therefore approximate value of $\sqrt{25.3}$ is $5 + 0.03 = 5.03$

9) If the radius of a sphere is measured as 7 m with an error of 0.02m, then find the approximate error in calculating its volume.

Ans: Given radius of the sphere $r = 7$m and $\Delta r = 0.02$ m.

Volume of sphere $V = (4/3)\ \pi\ r^3$.

Differentiate w.r.t 'r'

$dV/dr = (4/3)\ \pi(3r^2)$

Therefore $dV = (dV/dr)\ \Delta r$

$= (4\pi r^2)(\Delta r)$

$= (4\pi)\ (49)\ (0.02) \quad = 3.92\ \pi m^3$.

Therefore the approximate error in calculating the volume is $3.92\pi\ m^3$.

10) If the radius of sphere is measured as 9m with an error of 0.03m, then find the approximate error in calculating its surface area.

Ans: Radius of the sphere $r = 9$m, $\Delta r = 0.03$m.

Surface area of sphere $S = 4\pi r^2$.

Differentiate w.r.t. 'r'

$dS/dr = 4\pi(2r)$

Now $dS = (dS/dr)\ (\Delta r)$

$= (4\pi)(2)(r)\Delta r$

$=(8\pi)\ (9)\ (0.03)$

$=2.16\pi m^3$

THREE MARK QUESTIONS:

1) Find the local maxima and local minima if any, of the function $f(x) = x^2$ and also find the local maximum and local minimum values.

Ans: $f(x) = x^2$

Differentiate w.r.t. x

$f'(x) = 2x$

$f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x=0$

$f'(x) =2x$

$f''(x) = 2 > 0$

$\therefore$ By second derivative test x= 0 is a point of local minima .

$\therefore$ local minimum m value = $f(0) = 0^2 = 0$

2) Find the local maxima and local minima if any, of the function $f(x) =x^3 – 6x^2 + 9x + 15$ and also find the local maximum and local minimum values.

Ans: $f(x) = x^3- 6x^2+9x+15$

Differentiate w.r.t. x

$f'(x) = 3x^2-12x+9$

$f''(x) = 6x-12$

Now $f'(x) = 0 \Rightarrow 3x^2-12x+9=0$

$X^2-4x+3=0$

$(x-3)(x-1) = 0$

$\Rightarrow x=3, \quad x=1$

Now $f''(3) = 6(3)- 12 = 6>0$

$f''(1) = 6(1) – 12 = -6<0$

$\therefore$ By second derivative test, x= 3 is a point of local minima and x = 1 is a point of local maxima

$\therefore$ local maximum value = f(1) = (1)3-6(1)2+9(1)+15 =19

Local minimum value = f(3) = (3)3-6(3)2+9(3)+15=15.

3) Prove that the function f(x) = logx do not have maxima or minima.

Ans: $f(x) = \log x$

Differentiate w.r.t. x

$f'(x) = 1/x$

$f''(x) = -1/x^2$

Now $f'(x) = 0 \Rightarrow 1/x = 0$

$\Rightarrow x = \infty$

$\therefore$ The function do not have maxima or minima.

4) Prove that the function $f(x) = x^3 + x^2 + x + 1$ do not have maxima or minima.

Ans: $f(x) = x^3+x^2+x+1$

Differentiate w.r.t. x

$f'(X) = 3X^2+2X+1$

$f''(x) = 6x+2$

Now $f'(x) = 0 \Rightarrow 3x^2+2x+1=0$

$X = [-2\pm\sqrt{4-4(3)(1)}]/2(3)$

$X = [-2 \pm \sqrt{-8}]/6$ which is imaginary

$\therefore$ The given function do not have maxima or minima for all reals.

5) Find the absolute maximum value and the absolute minimum value of the function $f(x) = \sin x + \cos x$, $x \in [0,\pi]$.

Ans: $f(x) = \sin x + \cos x$,

Differentiate w.r.t. x

$f'(x) = \cos x - \sin x$

Now $f'(x) = 0$

$\cos x - \sin x = 0$

$\Rightarrow \sin x = \cos x \quad \therefore \tan x = 1$

$\Rightarrow x = \pi/4$ and $5\pi/4$

Now the value of the function f(x) at $x = \pi/4, 5\pi/4$ and end points of intervals that is 0 and π is

$f(0) = \sin 0 + \cos 0 = 0+1 = 1$

$f(\pi/4) = \sin(\pi/4) + \cos(\pi/4) = 1/\sqrt{2} + 1/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$

$f(5\pi/4) = \sin(5\pi/4) + \cos(5\pi/4) = (-1/\sqrt{2}) + (-1/\sqrt{2}) = -2/\sqrt{2} = -\sqrt{2}$

$f(\pi) = \sin\pi + \cos\pi = 0+(-1) = -1$

$\therefore$ Absolute maximum value of f on $[0,\pi]$ is $\sqrt{2}$ occurring at $x=\pi/4$.

$\therefore$ Absolute minimum value of f on $[0,\pi]$ is $-\sqrt{2}$ occurring at $x=5\pi/4$.

6) Find two numbers whose sum is 24 and whose product is as large as possible.

Ans: Let the numbers be 'x' & 'y'

Given $S = x+y = 24$

$\Rightarrow y = 24-x$

Product of numbers, P= x y is large

$P = x(24-x) = 24x-x^2$

Differentiate w.r.t. x

$dP/dx = 24-2x$

Differentiate w.r.t. x

$d^2P/dx^2 = -2<0$ Product is maximum

For the product to be maximum $dP/dx = 0$

$24-2x = 0 \Rightarrow x = 12$

$\therefore$ The numbers are x & 24-x,

12 & 24-12

12 & 12

$\therefore$ The numbers are 12 & 12

7) Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

Ans: Let numbers be x and y

Sum = $x+y = 16 \Rightarrow y = 16-x$

Given $S = x^3 + y^3$ is minimum

$= x^3 + (16-x)^3$

Differentiate w.r.t. x

$dS/dx = 3x^2 + 3(16-x)^2(-1)$

$d^2S/dx^2 = 6x - 3(2)(16-x)(-1)$

$=6x+6(16-x)$

For S to be minimum $dS/dx = 0$

$3x^2-3(16-x)^2 = 0$

$\Rightarrow x^2-(16-x)^2 = 0$

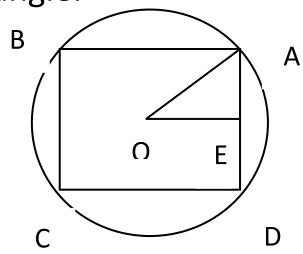
$\Rightarrow 32x - 256 = 0$

$\Rightarrow x = 8 \quad \therefore y = 16-x = 16-8 = 8$

Hence the numbers are 8 and 8.

8) Show that of all rectangles inscribed in a given fixed circle, the squares has the maximum area.

Ans : Let 'r' be the radius of circle ABCD is a rectangle.



OA = r , OE = x , AE = y ,In Δ le OAE ,

$OA^2 = OE^2 + AE^2$

$r^2 = x^2 + y^2$

$y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$

Area of rectangle A = x. y

$= x\sqrt{r^2 - x^2}$

Squaring both sides $A^2 = x^2(r^2 - x^2)$

Let $A^2 = B$ $\quad B = x^2(r^2 - x^2)$

Differentiate w.r.t. x

$dB/dx = x^2(-2x) + (r^2 - x^2)(2x)$

$= 2x(r^2 - 2x^2)$

$d^2B/dx^2 = 2x(-4x) + (r^2 - 2x^2)(2)$

$= 2r^2 - 12x^2$

For the area to be maximum dB/dx = 0

$2x(r^2 - 2x^2) = 0 \Rightarrow x = 0$ & $x^2 = r^2/2 \Rightarrow x = r/\sqrt{2}$

$d^2B/dx^2|_{x=r/\sqrt{2}} = 2r^2 - 12(r^2/2) = -4r^2 < 0$

$\therefore$ Area is maximum

$Y^2 = r^2 - x^2 = r^2 - r^2/2 = r^2/2$

Since $x = y = r/\sqrt{2}$, ABCD is a square.

9) Find the equation of the normal at the a point (am^2, am^3) for the curve $ay^2 = x^3$.

Ans: $ay^2 = x^3$

$Y^2 = x^3/a$

Differentiate w.r.t. x

$2y\ dy/dx = [1/a]\ 3x^2$

$dy/dx = 3x^2/2ay$

Slope of tangent $= dy/dx|_{(am^2, am^3)} = 3(am^2)^2 / 2a(am^3)$

$= 3a^2m^4/2a^2m^3 = 3m/2$

$\therefore$ slope of normal = $-2/3m$

$\therefore$ Equation of normal at (am^2, am^3) having slope $-2/3m$ is

$Y - am^3 = (-2/3m)(x - am^2)$.

10) Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

Ans: $y = x^3+2x+6$

Differentiate w.r.t. x

$dy/dx = 3x^2+2$ = slope of tangent

$\therefore$ slope of normal $= -1/(3x^2+2)$

Normal is parallel to $x + 14y + 4 = 0$

Slope of normal = slope of $x + 14y + 4 = 0$

$-1/(3x^2+2) = -1/14$

$3x^2+2 = 14$

$3x^2 = 12 \Rightarrow x = \pm 2$

When $x = 2$, $y = (2)^3 + 2(2) + 6 = 18$, $(2,18)$

When $x = -2$, $y = (-2)^3 + 2(-2)+6 = -6$, $(-2,-6)$

Slope of normal = $-1/14$

$\therefore$ equation of normal at $(2,18)$ is $y - 18 = (-1/14)(x - 2)$

$\Rightarrow x+14y - 254 = 0$

Also equation of normal at $(-2,-6)$ is $y+6 = (-1/14)(x+2)$

$\Rightarrow x + 14y+86 = 0$.

11) Find the points on the curve $x^2/9 + y^2/16 = 1$ at which the tangents are parallel to y axis.

Ans: $x^2/9 + y^2/16 = 1$

Differentiate w.r.t. x

$(1/9)\, 2x + (1/16)\, 2y\, (dy/dx) = 0$

$dy/dx = (-2x/9)/(y/8) = -16x/9y$

Tangent parallel to y axis.

Slope of tangent = slope of y axis

$-16x/9y = 1/0$

$\Rightarrow y = 0$

When $y = 0$, $x^2/9 + 0/16 = 1 \Rightarrow x^2 = 9$, $x = \pm 3$

$\therefore$ The points are $(\pm 3,0)$.

12) Find the equation of all lines having slope two which are tangents to the curve $y = 1/(x-3)$, $x \neq 3$.

Ans: $y = 1/(x-3)$

Differentiate w.r.t. x

$dy/dx =$

Given $dy/dx = 2$

$-1/(x-3)^2 = 2$

$2(x-3)^2 = -1$

$2(x^2-6x+9) = -1$

$2x^2-12x+19 = 0$

$X = (12 \pm \sqrt{144-152}) / 2(2)$ which is complex

$\therefore$ No tangent to the curve which has slope two.

13) Prove that the function 'f' given by $f(x) = \log(\sin x)$ is strictly increasing on $(0,\pi/2)$ and strictly decreasing on $(\pi/2,\pi)$.

Ans: $f(x) = \log(\sin x)$

Differentiate w.r.t. x

$f'(x) = (1/\sin x)(\cos x)$

$= \cot x$

Since for each $x \in (0,\pi/2)$, $\cot x > 0$ $\therefore$ $f'(x) > 0$

So f is strictly increasing in $(0,\pi/2)$

Since for each $x \in (\pi/2,\pi)$, $\cot x < 0$ $\therefore$ $f'(x) < 0$

So f is strictly decreasing in $(\pi/2,\pi)$.

FIVE MARKS QUESTIONS:

1) The volume of a cube is increasing at the rate of 8 c.c/s. How fast is the surface area increasing when the length of an edge is 12cm?

Ans: Let x , V, S be the length of side , volume and surface area of the cube respectively.

Given $dV/dt = 8$c.c/s, $dS/dt = ?$ when $x = 12$cm

Volume of cube $= V = x^3$

Differentiate w.r.t. t

$dV/dt = 3x^2 \cdot dx/dt$

$\Rightarrow 8 = 3(12)^2\ dx/dt$

$\Rightarrow dx/dt = 8/3(144) = 1/54$

Surface area of a cube $S = 6x^2$

Differentiate w.r.t. t

$dS/dt = 6(2x)\ (dx/dt)$

$= 12(12)\ (1/54) = 24/9 = 2.6$ sq.cm/s

$\therefore$ surface area of a cube is increasing at the rate of 2.6 sq.cm/s.

2) A stone is dropped into a quiet lake and waves in circles at the speed of 5cm/s. At the instant when the radius of circular wave is 8 cm, how fast is the enclosed area is increasing?

Ans: Let r , A be the radius and Area of a circle respectively

Given $dr/dt = 5$cm/s $dA/dt = ?$ when $r = 8$cm

Area of a circle $A = \pi r^2$

Differentiate w.r.t. t

$dA/dt = \pi\, 2r\ dr/dt$

$= \pi\, 2(8).(5)$

$= 80\pi\ cm^2/s$

$\therefore$ The enclosed area is increasing at the rate of $80\pi\ cm^2/s$ when $r = 8$cm.

3) The length 'x' of a rectangle is decreasing at the rate of 5cm/m and the width 'y' increasing at the rate of 4cm/m. When x = 8cm and y = 6cm, find the rates of changes of

(a) the perimeter and (b) the area of the rectangle.

Ans: Since the length 'x' is decreasing and width 'Y 'is increasing with respect to time,

we have $dx/dt = -5$ cm / m , $dy/dt = 4$ cm /m

(a) The perimeter P of a rectangle is given by

$P = 2(x+y)$

Differentiate w.r.t. t

$dP/dt = 2dx/dt + 2\ dY/dt$

$= 2(-5) + 2(4)$

$= -2$ cm/min

(b) The area 'A' of the rectangle is given by $A = x.y$

$\therefore dA/dt = dx/dt\ y + x\ dy/dt$

$= (-5)\ (6) + (8)\ (4)$

$= 2cm^2/m$

$\therefore$ The perimeter and area of a rectangle is decreasing and increasing at the rate of 2cm/m and 2 cm^2/m respectively .

4) A balloon, which always remains spherical, has a variable diameter (3/2)(2x+1). Find the rate of change of its volume w.r.t. x

Ans: Volume of a sphere $= V = (4/3)\pi r^3$

Given $2r = (3/2)(2x+1) \Rightarrow r = (3/4)(2x+1)$

$\therefore V = (4/3)\pi[(3/4)(2x+1)]^3$

$= (4/3)\pi (27/64)(2x+1)^3$

$V = (9\pi/16)(2x+1)^3$

Differentiate w.r.t. x

$dV/dx = (9\pi/16)\ 3(2x+1)^2(2)$

$= (27\pi/8)(2x+1)^2$

$\therefore$ volume of a sphere increases at the rate of $(27\pi/8)(2x+1)^2$

5) A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the radius is 10 cm.

Ans: Let 'r' and ' V' be the radius and volume of a sphere.

To find dV/dr = ? when r = 10cm

Volume of a sphere $V = (4/3)\pi r^3$

Differentiate w.r.t. r

$dV/dr = (4/3)\pi\ 3r^2$

$= 4\pi(10)^2$

$= 400\pi\ cm^3/cm$

$\therefore$ The volume of the spherical balloon is increasing with radius is $400\pi\ cm^3/cm$.

6) A water tank has the slope of an inverted right circular cone with its axis vertical and lower most. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubicmeter per hour. Find the rate at which the level of water is rising at the instant when the depth of water in the tank is 4m.

Ans: Let r, h and α be the radius , height and semi-vertical angle of cone.

Tan $\alpha = r/h \Rightarrow \alpha = \tan^{-1}(r/h)$

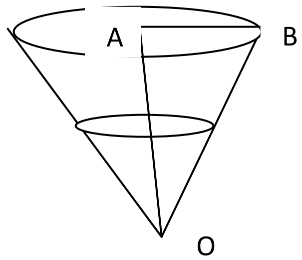
Given $\alpha = \tan^{-1}(0.5)$

$r/h = 0.5 = ½ \Rightarrow r = h/2$ Given $dV/dt = 5$ c.m/h

volume of a cone $V = (1/3)\pi r^2 h$: $dh/dt = ?$ when $h = 4$ m

$= (1/3)\pi (h^2/4) h$

$= (\pi/12).h^3$



OA = h

AB = r

Differentiate w.r.t. t

$dV/dt = (\pi/12).3h^2(dh/dt)$

$5 = (\pi/4)(4)^2(dh/dt)$

$\Rightarrow (dh/dt) = 5/4\pi = 35/88$ m/h $(\pi = 22/7)$

$\therefore$ Rate of change of water level = (35/88) m/h.

7) A ladder 5m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall at the rate of 2cm/s. How fast is its height of the ladder decreasing when the foot of the ladder is 4cm away from the wall?

Ans: Let AB be the ladder, AC wall, BC ground.

Let BC = x, AC = y

Given: AB = 5m, dx/dt = 2cm/s, dy/dt = ? when x = 4m.

From the fig, $x^2 + y^2 = 5^2$

$(4)^2 + y^2 = 25$

$y^2 = 9, \Rightarrow y = 3.$

Consider $x^2 + y^2 = 5^2$

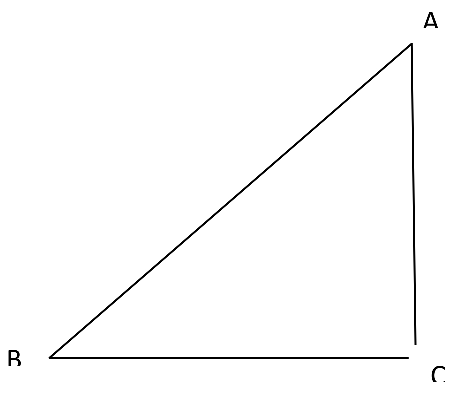
Differentiate w.r.t 't'

$2x(dx/dt) + 2y(dy/dt) = 0$

$2(4)(2) + 2(3)(dy/dt) = 0$

$6(dy/dt) = -16$

$dy/dt = -8/3.$



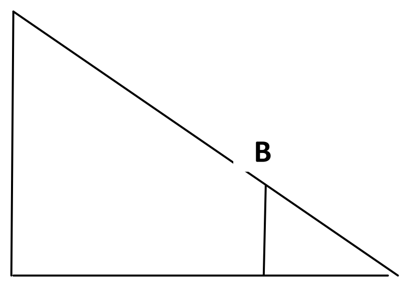
$\therefore$ Height of the ladder is decreasing at the rate of 8/3 cm/s.

8) A man 6ft tall moves away from a source of light 20ft above the ground level, his rate of walking being 4 m.p.h. At what rate is the length of his shadow changing? At what rate is the tip of his shadow moving?

Ans: At any time t, let AB = 6ft be the position of the man. Let C be the source of light. OC = 20 ft. Then AD is the shadow and D is the tip of the shadow.

Let OA = x and AD = y (be measured in miles)

Given: dx/dt = 4 m.p.h; dy/dt = ?; d(x+y)/dt = ?



From the figure, $\frac{OC}{AB} = \frac{OD}{AD}$, $\frac{20}{6} = \frac{X+Y}{Y}$

$\Rightarrow$ 20y = 6x+6y

$\Rightarrow$ 14y = 6x ; y = $\frac{3x}{7}$

Differentiate w.r.t 't'

$$\frac{dy}{dt} = \frac{3}{7}\frac{dx}{dt} = \frac{3}{7}.4 = \frac{12}{7}$$

$\therefore$ The shadow is changing at the rate of $\frac{12}{7}$ m.p.h.

Now $\frac{d(x+y)}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = 4 + \frac{12}{7} = \frac{40}{7}$

Therefore tip of the shadow is changing at the rate of $\frac{40}{7}$ m.p.h.

9) A stone is dropped into a pond, waves in the form of circles are generated and the radius of the outer most ripple increases at the rate of 2 inches/sec. How fast is the area increasing when the radius is 5 inches?

Ans: Let 'r' and 'A' be the radius and area of the circle respectively.

Given: $\frac{dr}{dt} = 2inches/sec, \frac{dA}{dt} = ?\, when\ r = 5inches$

Area of circle, $A = \pi r^2$Differentiate w.r.t. t

$$\frac{dA}{dt} = \pi.2r.\frac{dr}{dt}$$

$\frac{dA}{dt} = \pi.2(5)(2) = 20\pi$ sqinches/sec.

Therefore area of the circle increases at the rate of 20π sq. inches/sec.