# Question Bank

## 5. CONTINUITY AND DIFFERRENTIABILITY

### ONE MARK QUESTIONS

1. Find the derivative of $\cos(x^2)$ with respect to x.

Sol: let $y = \cos(x^2)$

$$\frac{dy}{dx} = -\sin(x^2)\frac{d}{dx}(x^2) = -2x\sin(x^2)$$

2. Find the derivative of $e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$ with respect to x.

Sol : let $y = e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$

$$\frac{dy}{dx} = e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

3. Find the derivative of log(logx) with respect to x.

Sol : $y = \log(\log x)$

$$\frac{dy}{dx} = \frac{1}{\log x}\frac{d}{dx}(\log x) = \frac{1}{x\log x}$$

4. Find the derivative of cos(sinx) with respect to x.

Sol : $y = \cos(\sin x)$

$$\frac{dy}{dx} = -\sin(\sin x)\frac{d}{dx}(\sin x) = -\cos x\sin(\sin x)$$

5. Find the derivative of $\sec(\tan\sqrt{x})$ with respect to x

Sol : $y = \sec(\tan\sqrt{x})$

$$\frac{dy}{dx} = \sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)\frac{d}{dx}(\tan\sqrt{x}) = \frac{\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)\sec^2(\sqrt{x})}{2\sqrt{x}}$$

6. Find the derivative of the function $\cos(\sqrt{x})$ with respect to x.

Sol : $y = \cos\left(\sqrt{x}\right)$

$$\frac{dy}{dx} = -\sin\left(\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{x}\right) = \frac{-\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

7. If $y = 3e^{2x} + 2e^{3x}$ find $\frac{dy}{dx}$

Sol : $y = 3e^{2x} + 2e^{3x}$

$$\frac{dy}{dx} = 3\frac{d}{dx}\left(e^{2x}\right) + 2\frac{d}{dx}\left(e^{3x}\right) = 6e^{2x} + 6e^{3x} = 6(e^{2x} + e^{3x})$$

8. Find the derivative of 5cosx – 3sinx with respect to x.

Sol : $y = 5\cos x - 3\sin x$

$$\frac{dy}{dx} = -5\sin x - 3\cos x$$

9. The function $f(x) = \frac{1}{x-5}$ is not continuous at x = 5. Justify the statement.

Sol : $f(x) = \frac{1}{x-5}$ is a quotient function. The function f(x) is not defined at x = 5 because

$f(5) = \frac{1}{5-5} = \frac{1}{0}$ is not defined. Therefore f(x) is continuous for all values of x except x = 5.

10. Find $\frac{dy}{dx}$ if $x - y = \pi$

Sol : $x - y = \pi$

$$\frac{d}{dx}\left(x - y\right) = \frac{d}{dx}\left(\pi\right)$$

$$\frac{d}{dx}(x) - \frac{d}{dx}(y) = 0$$

$$\frac{dy}{dx} = 1$$

11. If $y = \tan(2x+3)$ find $\frac{dy}{dx}$

Sol : $y = \tan(2x+3)$

$$\frac{dy}{dx} = \sec^2(2x+3)\frac{d}{dx}\left(2x+3\right) = 2\sec^2(2x+3)$$

12. Find the derivative of f given by $f(x) = \tan^{-1} x$ assuming it exists.

Sol : $y = \tan^{-1} x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+x^2}$

13. Prove that the function $f(x) = x^n$ is continuous at x = n, where n is a positive integer.

Sol : $f(x) = x^n$, $n \in N$. Here, f(x) is a polynomial function and $D_f = R$

$$\lim_{x \to n} f(x) = \lim_{x \to n} x^n = n^n = f(n).$$

Therefore f(x) is continuous at $n \in N$.

14. Find the derivative of $e^{\sin^{-1} x}$ with respect to x.

Sol : $y = e^{\sin^{-1} x}$

$$\frac{dy}{dx} = e^{\sin^{-1} x} \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

15. Find $\frac{dy}{dx}$, if $x = at^2$, $y = 2at$.

Sol : $x = at^2$, $y = 2at$

$$\frac{dx}{dt} = 2at, \qquad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

**TWO MARK QUESTIONS:**

1. Examine whether the function f given by $f(x) = x^2$ is continuous at x = 0.

Sol : $f(x) = x^2$ at x = 0; $f(0) = 0$.

Then $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0$

$$\lim_{x \to 0} f(x) = 0 = f(0).$$

$\therefore$ f is continuous at x = 0.

2. Discuss the continuity of the function f defined by $f(x) = \frac{1}{x}$, $x \neq 0$.

Sol : Fix any non zero real number c, we have $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$

Also, since for $c \neq 0$, $f(c) = \frac{1}{c}$, we have $\lim_{x \to c} f(x) = f(c)$ and hence, f is continuous at every point in the domain of f. Thus f is continuous function.

3. Find the derivative of the function $y = \frac{e^x}{\sin x}$ with respect to x.

Sol : $y = \frac{e^x}{\sin x}$

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}\left(e^x\right) - e^x \frac{d}{dx}\left(\sin x\right)}{\sin^2 x}$$

$$\frac{dy}{dx} = \frac{e^x \sin x - e^x \cos x}{\sin^2 x} = \frac{e^x\left(\sin x - \cos x\right)}{\sin^2 x}$$

4. Discuss the continuity of the function f given by $f(x) = x^3 + x^2 - 1$

Sol : Clearly f is defined at every real number c and its value at c is $c^3 + c^2 - 1$. We also know that

$$\lim_{x \to c} f(x) = \lim_{x \to c}\left(x^3 + x^2 - 1\right) = c^3 + c^2 - 1$$

Thus $\lim_{x \to c} f(x) = f(c)$, and hence f is continuous at every real number. This means f is a continuous function.

5. Verify Rolle's theorem for the function $y = x^2 + 2$, a = -2 and b = 2

Sol : The function $y = x^2 + 2$ is continuous in [- 2 , 2] and differentiable in (-2, 2). Also $f(-2) = f(2) = 6$ and hence the value of $f(x)$ at -2 and 2 coincide. $\therefore$ Rolle's theorem states that there is a point $c \in (-2, 2)$, where $f^{|}(c) = 0$. Since $f^{|}(x) = 2x$, we get c = 0. Thus at c = 0, we have $f^{|}(c) = 0$ and $c = 0 \in (-2, 2)$.

6. If f and g be two real functions continuous at real number c. Then show that f + g is continuous at x = c.

Sol : The continuity of f + g at x = c, clearly it is defined at x = c we have

$$\lim_{x \to c}\left(f + g\right)\left(x\right) = \lim_{x \to c}\left[f(x) + g(x)\right]$$

$$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c) = \left(f + g\right)(c)$$

Hence f + g is continuous at x = c.

7. Find $\frac{dy}{dx}$ if, $x = a\cos\theta$, $y = a\sin\theta$.

Sol : $x = a\cos\theta$, $y = a\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = a\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

8. Discuss the continuity of the function $f$ given by $f(x) = |x|$ at x = 0.

Sol : By definition $f(x) = \begin{cases} -x, & \text{if } x<0 \\ x, & \text{if } x \geq 0 \end{cases}$

Clearly the function is defined at x = 0 and f(0) = 0.

Let hand limit of f at x = 0 is $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = 0$

Right hand limit of f at x = 0 is $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$.

Thus the left hand limit, right hand limit and the value of the function coincide at x = 0.Hence, f is continuous at x = 0.

9. Find the derivative of the function $\frac{\sin(ax+b)}{\cos(cx+d)}$ with respect to x.

Sol : $y = \frac{\sin(ax+b)}{\cos(cx+d)}$

$$\frac{dy}{dx} = \frac{\cos(cx+d)\frac{d}{dx}\sin(ax+b) - \sin(ax+b)\frac{d}{dx}\cos(cx+d)}{\cos^2(cx+d)}$$

$$\frac{dy}{dx} = \frac{a\cos(cx+d)\cos(ax+b) + c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$$

10. Discuss the continuity of sine function.

Sol : $f(x) = \sin x$ is defined for every real number. Let c be a real number. Put x = c + h.

If $x \to c$ we know that $h \to 0$. Therefore

$\lim_{x\to c} f(x) = \lim_{x\to c} \sin x$

$$= \lim_{h \to 0} \sin(c+h) = \lim_{h \to 0} \left[\sin c.\cosh + \cos c.\sinh\right]$$

$$= \lim_{h \to 0} \sin c.\cosh + \lim_{h \to 0} \cos c.\sinh = \sin c + 0 = \sin c = f(c)$$

Thus $\lim_{x \to c} f(x) = f(c)$

Therefore f is a continuous function.

11. Differentiate $x^{\sin x} \quad x > 0$ with respect to x.

Sol : $y = x^{\sin x}$

Taking log on both sides

$$\log y = \log x^{\sin x}$$

$$\log y = \sin x \log x$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \frac{d}{dx}(\sin x)\log x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \cos x \log x$$

$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

12. Differentiate the function $\sin\left(\tan^{-1} e^{-x}\right)$ with respect to x.

Sol : $y = \sin\left(\tan^{-1} e^{-x}\right)$

$$\frac{dy}{dx} = \cos\left(\tan^{-1} e^{-x}\right)\frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$

$$\frac{dy}{dx} = \frac{-\cos\left(\tan^{-1} e^{-x}\right)e^{-x}}{1+e^{-2x}}$$

13. If $x = 2at^2$, $y = at^4$ find $\frac{dy}{dx}$

Sol : $x = 2at^2, \qquad y = at^4$

$$\frac{dx}{dt} = 4at, \qquad \frac{dy}{dt} = 4at^3$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at^3}{4at} = t^2$$

14. If $xy = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$

Sol : $xy = e^{x-y}$

Differentiating both sides with respect to x.

$$\frac{d}{dx}(xy) = \frac{d}{dx}\left(e^{x-y}\right)$$

$$x\frac{dy}{dx} + y = e^{x-y}\left(1 - \frac{dy}{dx}\right)$$

$$x\frac{dy}{dx} + y = e^{x-y} - e^{x-y}\frac{dy}{dx}$$

$$x\frac{dy}{dx} + e^{x-y}\frac{dy}{dx} = e^{x-y} - y$$

$$\left(x + e^{x-y}\right)\frac{dy}{dx} = e^{x-y} - y$$

$$\frac{dy}{dx} = \frac{e^{x-y} - y}{x + e^{x-y}} = \frac{xy - y}{x + xy} = \frac{y(x-1)}{x(y+1)}$$

15. If $y = \cos x \cos 2x \cos 3x$ find $\frac{dy}{dx}$

Sol : $y = \cos x \cos 2x \cos 3x$

Taking log on both sides, we get

$$\log y = \log(\cos x \cos 2x \cos 3x)$$

$$\log y = \log\cos x + \log\cos 2x + \log\cos 3x$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{\sin x}{\cos x} - \frac{2\sin 2x}{\cos 2x} - \frac{3\sin 3x}{\cos 3x}$$

$$\frac{dy}{dx} = -\cos x \cos 2x \cos 3x\left(\tan x + 2\tan 2x + 3\tan 3x\right)$$

16. If $\sqrt{x} + \sqrt{y} = a$ prove that $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

Sol : $\sqrt{x}+\sqrt{y}=a$

Differentiate w.r.t x we get

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{x}} \qquad \Rightarrow \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

17. Find the derivative of $(\sin x-\cos x)^{\sin x-\cos x}$ with respect to x.

Sol : $y=(\sin x-\cos x)^{(\sin x-\cos x)}$

Taking log on both sides

$$\log y=\log(\sin x-\cos x)^{(\sin x-\cos x)}$$

$$\log y=(\sin x-\cos x)\log(\sin x-\cos x)$$

Differentiate w.r.t x

$$\frac{1}{y}\frac{dy}{dx}=(\sin x-\cos x)\frac{d}{dx}\log(\sin x-\cos x)+\frac{d}{dx}(\sin x-\cos x)\log(\sin x-\cos x)$$

$$\frac{1}{y}\frac{dy}{dx}=(\sin x-\cos x)\frac{\cos x+\sin x}{\sin x-\cos x}+(\cos x+\sin x)\log(\sin x-\cos x)$$

$$\frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}(\cos x+\sin x)\{1+\log(\sin x-\cos x)\}$$

18. If $y=\left(\sin^{-1}x\right)^{x}$ find $\frac{dy}{dx}$

Sol : $y=\left(\sin^{-1}x\right)^{x}$

Taking logarithm on both sides

$$\log y=\log\left(\sin^{-1}x\right)^{x}$$

$$\log y=x\log\left(\sin^{-1}x\right)$$

$$\frac{1}{y}\frac{dy}{dx}=x\frac{d}{dx}\log\left(\sin^{-1}x\right)+\frac{d}{dx}(x)\log\left(\sin^{-1}x\right)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{x}{\sin^{-1}x\sqrt{1-x^{2}}}+\log\left(\sin^{-1}x\right)$$

$$\frac{dy}{dx}=y\left\{\frac{x}{\sin^{-1}x\sqrt{1-x^2}}+\log\left(\sin^{-1}x\right)\right\}=\left(\sin^{-1}x\right)^x\left\{\frac{x}{\sin^{-1}x\sqrt{1-x^2}}+\log\left(\sin^{-1}x\right)\right\}$$

19. If $y=\sin\left(\log_e x\right)$ prove that $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{x}$

Sol : $y=\sin\left(\log_e x\right)$

$$\frac{dy}{dx}=\cos\left(\log_e x\right)\frac{d}{dx}\left(\log_e x\right)=\frac{\cos\left(\log_e x\right)}{x}$$

$$\frac{dy}{dx}=\frac{\sqrt{1-\left(\sin(\log_e x)\right)^2}}{x}=\frac{\sqrt{1-y^2}}{x}$$

**THREE MARK QUESTIONS:**

1. If $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ 0 < x < 1 Find $\frac{dy}{dx}$

Sol : $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x=\tan\theta$

$$y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}\left(\cos 2\theta\right)$$

$$y=2\theta=2\tan^{-1}x$$

Differentiate w.r. t x

$$\frac{dy}{dx}=\frac{1}{\left(1+x^2\right)}$$

2. If $x^3+x^2y+xy^2+y^3=81$. Find $\frac{dy}{dx}$

Sol : $x^3+x^2y+xy^2+y^3=81$

Differentiate w.r. t x

$$3x^2+x^2\frac{dy}{dx}+2xy+x\left(2y\frac{dy}{dx}\right)+y^2+3y^2\frac{dy}{dx}=0$$

$$\left(x^2+2xy+3y^2\right)\frac{dy}{dx}=-\left(3x^2+2xy+y^2\right)$$

$$\frac{dy}{dx} = -\frac{\left(3x^2 + 2xy + y^2\right)}{x^2 + 2xy + 3y^2}$$

3. Diffentiate $y = \sqrt{\frac{(x-3)\left(x^2+4\right)}{3x^2+4x+5}}$ w.r. t x.

Sol : $y = \sqrt{\frac{(x-3)\left(x^2+4\right)}{3x^2+4x+5}}$

Taking logarithm on both sides, we have

$$\log y = \frac{1}{2}\left[\log(x-3) + \log\left(x^2+4\right) - \log\left(3x^2+4x+5\right)\right]$$

Differentiating on both sides w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-3)\left(x^2+4\right)}{3x^2+4x+5}}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

4. Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Sol : $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$$y = 2\theta = 2\tan^{-1} x$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

5. Differentiate the function $(\log x)^{\cos x}$ with respect to x.

Sol : $y=(\log x)^{\cos x}$

Taking logarithm on both sides

$$\log y=\log(\log x)^{\cos x}$$

$$\log y=\cos x\log(\log x)$$

Differentiating w.r. t. x on both sides, we get

$$\frac{1}{y}\frac{dy}{dx}=\cos x\frac{d}{dx}(\log\log x)+\frac{d}{dx}(\cos x)\log(\log x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{\cos x}{x\log x}-\sin x\log(\log x)$$

$$\frac{dy}{dx}=y\left[\frac{\cos x}{x\log x}-\sin x\log(\log x)\right]=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\log(\log x)\right]$$

6. Find $\frac{dy}{dx}$ if $y^x=x^y$

Sol : $y^x=x^y$

Taking logarithm on both sides

$$\log y^x=\log x^y$$

$$x\log y=y\log x$$

Differentiating with respect to x, on both sides, we get

$$x\frac{d}{dx}(\log y)+\log y\frac{d}{dx}(x)=y\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(y)$$

$$\frac{x}{y}\frac{dy}{dx}+\log y=\frac{y}{x}+\log x\frac{dy}{dx}$$

$$\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\frac{dy}{dx}=\frac{y(y-x\log y)}{x(x-y\log x)}$$

7. Differentiate $\sin^2 x$ with respect to $e^{\cos x}$

Sol : let $u=\sin^2 x$ and $v=e^{\cos x}$

Differentiate w.r.t x

$$\frac{du}{dx} = 2\sin x \cos x \qquad \text{and} \qquad \frac{dv}{dx} = -\sin x e^{\cos x}$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{2\sin x \cos x}{-\sin x e^{\cos x}} = -\frac{2\cos x}{e^{\cos x}}$$

8. Differentiate $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ with respect to x.

Sol : $y = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

$$y = \tan^{-1}\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right) = \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) = \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$y = \frac{x}{2}$$

Differentiating w.r.t x on both sides

$$\frac{dy}{dx} = \frac{1}{2}$$

9. Verify mean value theorem if $f(x) = x^3 - 5x^2 - 3x$ in the interval [a, b] where a = 1 and b = 3. Find $c \in (1,3)$ for which $f^{|}(c) = 0$.

Sol: Given $f(x) = x^3 - 5x^2 - 3x \ \ x \in [1,3]$ which is a polynomial function.

Since a polynomial function is continuous and derivable at all $x \in R$

(1) f(x) is continuous on [1,3] (2) f(x) is derivable on (1, 3)

Therefore condition of mean value theorem satisfied on [1,3]. Hence, $\exists$ at least one real $c \in (1,3)$

$$f^{|}(c) = \frac{f(3) - f(1)}{3-1} = \frac{3^3 - 5(3)^2 - 3(3) - [1 - 5(1) - 3(1)]}{2}$$

$$f^{|}(c) = -\frac{20}{2} = -10$$

$f^{|}(x) = 3x^2 - 10x - 3;$ $\qquad$ $f^{|}(c) = 3c^2 - 10c - 3 = -10$

$$3c^2 - 10c + 7 = 0$$

$$3c^2 - 7c - 3c + 7 = 0;$$

$c(3c-7)-(3c-7)=0 \Rightarrow$ c = $1 \notin (1,3)$ $\quad c=\frac{7}{3} \in (1,3)$.

Hence the mean theorem satisfied for given function in the given interval.

10. If $y=\cos^{-1}x$ find $\frac{d^2y}{dx^2}$ in terms of y alone.

Sol : $y=\cos^{-1}x$

$x=\cos y$

Differentiating w.r.t y, we get

$$\frac{dx}{dy}=-\sin y$$

$$\frac{dy}{dx}=-\cos ecy$$

Again differentiating w.r. t x , we get

$$\frac{d^2y}{dx^2}=\cos ecy\cot y\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}=-\cos ec^2y\cot y$$

11. Find the derivative of $(\log x)^{\log x}$ with respect to x.

Sol : $y=(\log x)^{\log x}$

Taking logarithm on both sides

$$\log y=\log(\log x)^{\log x}$$

$$\log y=\log x\log(\log x)$$

Differentiating w.r. t x on both sides

$$\frac{1}{y}\frac{dy}{dx}=\log x\frac{d}{dx}(\log(\log x))+\log(\log x)\frac{d}{dx}(\log x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{\log x}{x\log x}+\frac{\log(\log x)}{x}$$

$$\frac{dy}{dx}=y\left[\frac{1}{x}+\frac{\log(\log x)}{x}\right]=(\log x)^{\log x}\left[\frac{1}{x}+\frac{\log(\log x)}{x}\right]$$

12. Find $\frac{dy}{dx}$ if $y = \cos x^3.\sin^2\left(x^5\right)$

Sol : $y = \cos\left(x^3\right)\sin^2\left(x^5\right)$

Differentiating w.r.t x on both sides

$$\frac{dy}{dx} = \cos x^3 \frac{d}{dx}\left(\sin^2\left(x^5\right)\right) + \sin^2\left(x^5\right)\frac{d}{dx}\left(\cos x^3\right)$$

$$\frac{dy}{dx} = \cos x^3\left(2\sin x^5 \cos x^5\right)5x^4 + \sin x^5\left(-\sin x^3\right)3x^2$$

$$\frac{dy}{dx} = 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^5 \sin x^3$$

13. Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8,\ x \in [-4, 2]$.

Sol : Given $f(x) = x^2 + 2x - 8,\ x \in [-4, 2]$. Since a polynomial function is continuous and derivable on R. (1) f(x) is continuous on [-4,2]  (2) f(x) is derivable on [-4,2].

Also $f(-4) = (-4)^2 + 2(-4) - 8 = 0$ and $f(2) = 2^2 + 2\times 2 - 8 = 0 \Rightarrow f(-4) = f(2)$.

This means that all the conditions of Rolle's theorem are satisfied by f(x) in [-4,2].

Therefore there exist at least one real number $c \in (-4, 2)$ such that $f^{|}(c) = 0$.

$$f(x) = x^2 + 2x - 8 \Rightarrow f^{|}(x) = 2x + 2$$

$$f^{|}(c) = 0 \Rightarrow 2c + 2 = 0 \Rightarrow c = -1 \in (-4, 2)$$

$\therefore$ Rolle's theorem is verified with c = - 1

14. If $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$ then prove that $\frac{dy}{dx} = -\frac{y}{x}$

Sol : $x = \sqrt{a^{\sin^{-1} t}}$  $y = \sqrt{a^{\cos^{-1} t}}$

$x = a^{\frac{1}{2}\sin^{-1} t}$  $y = a^{\frac{1}{2}\cos^{-1} t}$

Differentiating w.r.t "t" we get

$$\frac{dx}{dt} = a^{\frac{1}{2}\sin^{-1} t} \log a \frac{d}{dx}\left(\frac{1}{2}\sin^{-1} t\right) \qquad \frac{dy}{dt} = a^{\frac{1}{2}\cos^{-1} t} \log a \frac{d}{dx}\left(\frac{1}{2}\cos^{-1} t\right)$$

$$\frac{dx}{dt} = \frac{a^{\frac{1}{2}\sin^{-1} t} \log a}{2\sqrt{1-t^2}} \qquad \frac{dy}{dt} = -\frac{a^{\frac{1}{2}\cos^{-1} t} \log a}{2\sqrt{1-t^2}}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-a^{\frac{1}{2}\cos^{-1}t}\times\frac{\log a}{2\sqrt{1-t^2}}}{a^{\frac{1}{2}\sin^{-1}t}\times\frac{\log a}{2\sqrt{1-t^2}}}=-\frac{\sqrt{a^{\sin^{-1}t}}}{\sqrt{a^{\cos^{-1}t}}}$$

$$\frac{dy}{dx}=-\frac{y}{x}$$

15. Find the derivative of $x^x-2^{\sin x}$ with respect to x.

Sol : let $y=x^x-2^{\sin x}=u-v$

Where $u=x^x$ and $v=2^{\sin x}$

Taking log on both sides

$\log u=\log x^x$ and $\log v=\log 2^{\sin x}$

$\log u=x\log x$ and $\log v=\sin x\log 2$

Differentiate with respect to x we get

$\frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}(\log x)+\frac{d}{dx}(x)\log x$ and $\frac{1}{v}\frac{dv}{dx}=\cos x\log 2$

$\frac{du}{dx}=u\left[\frac{x}{x}+1.\log x\right]=x^x\left[1+\log x\right]$ and $\frac{dv}{dx}=v\cos x\log 2=2^{\sin x}\cos x\log 2$

$y=u-v$

$$\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}=x^x\left[1+\log x\right]-2^{\sin x}\cos x\log 2$$

16. If $x=a(\theta+\sin\theta)$ and $y=a(1-\cos\theta)$ prove that $\frac{dy}{dx}=\tan\frac{\theta}{2}$

Sol : $x=a(\theta+\sin\theta)$ $y=a(1-\cos\theta)$

Differentiating w.r.t x on both sides

$\frac{dx}{d\theta}=a(1+\cos\theta)$ $\frac{dy}{d\theta}=a(0+\sin\theta)=a\sin\theta$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a\sin\theta}{a(1+\cos\theta)}\Rightarrow\quad\frac{dy}{dx}=\frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2a\cos^2\frac{\theta}{2}}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\tan\frac{\theta}{2}$$

17. If a function f(x) is differentiable at x = c , prove that it is continuous at x = c.

Sol : Since f is differentiable at c, we have $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f^{|}(c)$

But for $x\neq c$, we have $f(x)-f(c)=\frac{f(x)-f(c)}{x-c}.\left(x-c\right)$

Therefore $\lim_{x\to c}[f(x)-f(c)]=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}.\left(x-c\right)\right]$

Or $\lim_{x\to c}f(x)-\lim_{x\to c}f(c)=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right]\lim_{x\to c}\left(x-c\right)$

$$= f^{|}(c)\ .\ 0 = 0$$

$\lim_{x\to c}f(x)=f(c)$ . Hence f is continuous at x = c.

**FIVE MARK QUESTIONS**

1. If $y=3\cos\left(\log x\right)+4\sin\left(\log x\right)$ prove that $x^2y_2+xy_1+y=0$ .

Sol : $y=3\cos\left(\log x\right)+4\sin\left(\log x\right)$

Differentiating w.r.t x on both sides

$$y_1=-\frac{3\sin\left(\log x\right)}{x}+\frac{4\cos\left(\log x\right)}{x}$$

$$xy_1=-3\sin\left(\log x\right)+4\cos\left(\log x\right)$$

Again differentiating on both sides we get

$$xy_2+\left(1\right)y_1=-\frac{3\cos\left(\log x\right)}{x}-\frac{4\sin\left(\log x\right)}{x}$$

$$x^2y_2+xy_1=-\left[3\cos\left(\log x\right)+4\sin\left(\log x\right)\right]$$

$$x^2y_2+xy_1=-y \quad \Rightarrow x^2y_2+xy_1+y=0$$

2. If $y=3e^{2x}+2e^{3x}$ prove that $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$

Sol : $y=3e^{2x}+2e^{3x}$

$$\frac{dy}{dx}=6e^{2x}+6e^{3x}=6\left(e^{2x}+e^{3x}\right)$$

$$\therefore\frac{d^2y}{dx^2}=12e^{2x}+18e^{3x}=6\left(2e^{2x}+3e^{3x}\right)$$

Hence $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=6(2e^{3x}+3e^3x)-30\left(e^{2x}+e^{3x}\right)+6\left(3e^{2x}+2e^{3x}\right)$

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=12e^{3x}+18e^3x-30e^{2x}-30e^{3x}+18e^{2x}+12e^{3x}=0$$

3. If $y=\left(\tan^{-1}x\right)^2$ prove that $\left(x^2+1\right)^2 y_2+2x\left(x^2+1\right)y_1=2$.

Sol : $y=\left(\tan^{-1}x\right)^2$

Differentiating w.r.t on both sides

$$y_1=\frac{2\tan^{-1}x}{1+x^2}$$

$$\Rightarrow\left(1+x^2\right)y_1=2\tan^{-1}x$$

Again differentiating w.r.t x on both sides

$$\left(1+x^2\right)y_2+2xy_1=\frac{2}{1+x^2}$$

On cross multiplication, we get

$$\left(1+x^2\right)^2 y_2+2x\left(1+x^2\right)y_1=2$$

4. If $y=Ae^{mx}+Be^{nx}$, Show that $\frac{d^2y}{dx^2}-\left(m+n\right)\frac{dy}{dx}+mny=0$.

Sol : $y=Ae^{mx}+Be^{nx}$

Differentiating w.r.t x on both sides

$\frac{dy}{dx}=Ame^{mx}+Bne^{nx}$ Again differentiate w.r.t x on both sides

$$\frac{d^2y}{dx^2}=Am^2e^{mx}+Bn^2e^{nx}$$

Hence $\frac{d^2y}{dx^2}-\left(m+n\right)\frac{dy}{dx}+mny=m^2Ae^{mx}+n^2Be^{nx}-\left(m+n\right)\left(Ame^{mx}+Bne^{nx}\right)+mny$

$$=m^2Ae^{mx}+n^2Be^{nx}-Am^2e^{mx}-Bmne^{nx}-Amne^{mx}-n^2Be^{nx}+mny$$

$$=-Bmne^{nx}-Amne^{mx}+mny=-mn\left(Ae^{mx}+Be^{nx}\right)+mny$$

$$=-mny+mny=0 \qquad \left(\because y=Ae^{mx}+Be^{nx}\right)$$

5. If $y=\sin^{-1}x$ prove that $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$

Sol : $y=\sin^{-1}x$

Differentiate w.r.t x, we get

$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$ On cross multiplication

$\sqrt{1-x^2}\,\frac{dy}{dx}=1$

Again Differentiate w.r.t x, we get

$\sqrt{1-x^2}\,\frac{d^2y}{dx^2}-\frac{2x}{2\sqrt{1-x^2}}\frac{dy}{dx}=0$

Taking Lcm and simplifying, we get

$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$

6. If $y=\cos^{-1}x$ prove that $\left(1-x^2\right)y_2-xy_1=0$

Sol : $y=\sin^{-1}x$

Differentiate w.r.t x, we get

$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$

On cross multiplication

$\sqrt{1-x^2}\,y_1=-1$

Again Differentiate w.r.t x, we get

$\sqrt{1-x^2}\,y_2-\frac{2x}{2\sqrt{1-x^2}}\,y_1=0$

Taking Lcm and simplifying, we get $\left(1-x^2\right)y_2-xy_1=0$

7. If $y=5\cos x-3\sin x$, prove that $\frac{d^2y}{dx^2}+y=0$

Sol : $y=5\cos x-3\sin x$

Differentiating w.r.t x , on both sides

$$\frac{dy}{dx} = -5\sin x - 3\cos x$$ Again differentiating w.r.t x we get

$$\frac{d^2y}{dx^2} = -5\cos x + 3\sin x = -\left(5\cos x - 3\sin x\right)$$

$$\frac{d^2y}{dx^2} = -y \qquad \Rightarrow \frac{d^2y}{dx^2} + y = 0$$

8. If $e^y\left(x+1\right)=1$, prove that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Sol : $e^y\left(x+1\right)=1$

Differentiate w.r.t x on both sides

$$e^y\frac{d}{dx}\left(x+1\right) + \left(x+1\right)\frac{d}{dx}\left(e^y\right) = 0$$

$$e^y + \left(x+1\right)e^y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{1}{x+1}$$ Again differentiate w.r.t x , we get

$$\frac{d^2y}{dx^2} = \frac{1}{\left(x+1\right)^2}$$

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

9. If $y = 500e^{7x} + 600e^{-7x}$ then show that $\frac{d^2y}{dx^2} = 49y$

Sol : $y = 500e^{7x} + 600e^{-7x}$

Differentiate w.r.t x

$$\frac{dy}{dx} = 500\left(7e^{7x}\right) + 600\left(-7e^{-7x}\right)$$

Again differentiate w.r.t x

$$\frac{d^2y}{dx^2} = 500\left(49e^{7x}\right) + 600\left(49e^{-7x}\right) \qquad \frac{d^2y}{dx^2} = 49\left(500e^{7x} + 600e^{-7x}\right)$$

$$\frac{d^2y}{dx^2} = 49y$$