Chapter - 04
Determinants

one - 1 mark question, one - 2 mark question
one - 4 mark question, one - 4 mark question.

One mark questions

1. Evaluate the following determinants

i) $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$ ii) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ iii) $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

iv) $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$ v) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ vi) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

vii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ viii) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ ix) $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

x) $\begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

Answer i) $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1)-4(-5)=18$

ii) $\cos^2\theta+\sin^2\theta=1$

iii) x^3-x^2+2

iv) 1

v) -12

vi) 46

vii) 0

viii) 5

ix) -52

x) 0

2\. i> If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

ii> If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$

iii> If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$

Answer i> $|A| = 2 - 8 = -6$

$$|2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 8 - 32 = -24 = 4(-6)$$

$$= 4|A|$$

ii> $|A| = 4$ (Try)

$|3A| = 108 = 27(4) = 27|A|$

iii> $|A| = 0$

3\. Find the values of x, if

i> $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

ii> $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

iii> $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

Answer i> $2 - 20 = 2x^2 - 24$

$24 - 18 = 2x^2$

$x^2 = 3$

$x = \pm\sqrt{3}$

ii> $x = 2$

iii> $x = \pm 2\sqrt{2}$.

3) Evaluate

i) $\begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$ ii) $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Answer

i) $\Delta = 0 \quad \because R_1 \cong R_3$

ii) $\Delta = \begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

$= 6 \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0 \quad \because R_1 \cong R_3$

TWO MARK QUESTIONS:

1) Prove the following properties

i) In case of third order determinant, the value of the determinant is unaltered if the rows are changed into columns and columns into rows

ii) If any two rows (or columns) of a determinant are interchanged, then sign of the determinant changes

iii) If any two rows (or columns) of a determinant are identical then the value of the determinant is zero.

Answer: (For proofs refer text books)

i) Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \rightarrow ①$

Now $\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \quad \rightarrow ②$$

From ① and ②, we get

$$\Delta = \Delta'$$

ii) Try yourself

iii) Try yourself.

2) Using the properties of determinants and without expanding prove that

i) $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

ii) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

iii) $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

iv) $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

v) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

vi) $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

vii) $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

3) ~~Refned~~

Answer

i) Opperate $C_3 \to C_3 - C_1 - C_2$

$$LHS = \begin{vmatrix} x & a & 0 \\ y & b & 0 \\ z & c & 0 \end{vmatrix}$$

$$= 0 = RHS$$

ii) Opperate $C_1 \to C_1 + C_2 + C_3$

$$LHS = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

$$= 0 = RHS$$

iii) Opperate $C_3 \to C_3 - 9C_2 - C_1$

$$LHS = \begin{vmatrix} 2 & 7 & 0 \\ 3 & 8 & 0 \\ 5 & 9 & 0 \end{vmatrix}$$

$$= 0 = RHS.$$

iv) $C_3 \to C_3 + C_2$

$$LHS = \begin{vmatrix} 1 & bc & ab+ca+bc \\ 1 & ca & bc+ab+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

$$= (ab+bc+ca) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$$= 0 \qquad \because C_1 \cong C_3.$$

v) $R_1 \to R_1 + R_2$

$$LHS = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0 \qquad \because \; R_1 \cong R_3$$

vi) $R_2 \to R_2 - R_1 - 2R_3$

$$LHS = \begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ x & y & z \end{vmatrix}$$

$$= 0 = RHS$$

3) Find the area of the triangle with vertices at the points in each of the following using determinants

i) $(1,0)$, $(6,0)$, $(4,3)$

ii) $(2,7)$, $(1,1)$, $(10,8)$

iii) $(-2,-3)$, $(3,2)$, $(-1,-8)$

Answer

i) $$\text{Area of } \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{15}{2} \text{ sq. units}$$

ii) $\Delta = \frac{47}{2}$ sq. units

iii) $\Delta = 15$ sq. units.

4) Show that the points $A(a, b+c)$, $B(b, c+a)$, $C(c, a+b)$ are collinear

Answer: $$\Delta ABC = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

Operate: $C_1 \to C_1 + C_2$

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}(a+b+c)\begin{vmatrix} 0 & b+c & 1 \\ 0 & c+a & 1 \\ 6 & a+b & 1 \end{vmatrix} = 0$$

$\therefore$ A, B and C are collinear.

5) Find equation of line joining (1,2) and (3,6) using determinants.

Answer. Let A(1, 2), B(3,6). be given pts

Let C(x, y) be any ~~pt~~ point on the line AB.

$\therefore$ A, B and C are collinear.

$\therefore$ $\Delta ABC = 0$

$$\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0$$

$$x(2-6) - y(1-3) + 1(6-6) = 0$$

$$-4x + 2y = 0$$

$$y = 2x.$$

6) Find the values of k if

i) area of triangle is 35 sq. units with vertices (2, -6), (5, 4) and (k, 4).

ii) area of triangle is 3 sq. units with vertices (1, 3), (0,0), (k, 0).

Answer i) Given $\frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = \cancel{35} \pm 35$

$$\Rightarrow 2(4-4) + 6(5-k) + 1(20-4k) = \pm 35 \times 2$$

$$30 - 6k + 20 - 4k = \pm 70$$

$$-10k = \pm 70 - 50$$

$\therefore$ $k = 12$ or $k = -2$.

ii) Try yourself $k = \pm 2$

7) Examine the consistency of the system of equations

i) $x + 2y = 2$
$2x + 3y = 3$

ii) $x + 3y = 5$
$2x + 6y = 8$

Answer

i) The given system is equivalent to $AX = B$
where $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$|A| = 3-4 = -1 \neq 0$

$\therefore$ A is non singular

$\therefore$ The given system of equation is consistent.

ii) The given system is equivalent to $AX = B$
where $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$|A| = 6-6 = 0$

$\text{adj} A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$\cancel{A(\text{adj}A) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}}$

$(\text{adj} A) B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30-18 \\ -10+8 \end{bmatrix}$

$= \begin{bmatrix} 12 \\ -2 \end{bmatrix} \neq 0$

$\therefore$ The given system of equations is inconsistent.

FOUR MARK QUESTIONS:

1) By using properties of determinants, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Answer:

Operate $C_1 \to C_1 - C_2$ ~~$C_3 \to C_3 - C_2$~~ and $C_2 \to C_2 - C_3$

$$LHS = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

Taking $a-b$ and $b-c$ common from C_1 and C_2

$$\therefore LHS = (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)\{b^2+bc+c^2-a^2-ab-b^2\}$$

$$= (a-b)(b-c)\{bc-ab+(c^2-a^2)\}$$

$$= (a-b)(b-c)\{b(c-a)+(c-a)(c+a)\}$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

$= RHS.$

2) By using the properties of determinants, show that

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Answer: Operate $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$LHS = \begin{vmatrix} x-y & x^2-y^2 & yz-zx \\ y-z & y^2-z^2 & zx-xy \\ z & z^2 & xy \end{vmatrix}$$

Taking common ~~from~~ $(x-y)$ and $(y-z)$ from R_1 and R_2 respectively

$$LHS = (x-y)(y-z)\begin{vmatrix} 1 & x+y & -z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Operate $R_2 \to R_2 - R_1$

$$LHS = (x-y)(y-z)\begin{vmatrix} 1 & x+y & -z \\ 0 & z-x & z-x \\ z & z^2 & xy \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)\begin{vmatrix} 1 & x+y & -z \\ 0 & 1 & 1 \\ z & z^2 & xy \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)\{1(xy-z^2) - 0 + z(x+y+z)\}$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx)$$

$= RHS.$

3) Prove by using properties of determinants

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

Answer: Operate $C_1 \to C_1 + C_2 + C_3$

$$LHS = \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix}$$

Taking $5x+4$ common from C_1.

$$LHS = (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$LHS = (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & 4-x & 0 \\ 0 & 0 & 4-x \end{vmatrix}$$

$$= (5x+4)(4-x)^2 = RHS.$$

4) By properties of determinants, show the following

i) $$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

ii) $$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

iii) $$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

iv) $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

v) $$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

vi) $$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$$

vii) $$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Answer: Try yourself

FIVE MARK QUESTIONS

~~Ex~~ Solve system of linear equations, using matrix method.

1) $2x+y+z=1$

$x-2y-z=\frac{3}{2}$

$3y-5z=9$

2) $x-y+z=4$

$2x+y-3z=0$

$x+y+z=2$

3) $2x+3y+3z=5$

$x-2y+z=-4$

$3x-y-2z=3$

4) $x-y+2z=7$

$3x+4y-5z=-5$

$2x-y+3z=12$

5) $3x-2y+3z=8$

$2x+y-z=1$

$4x-3y+2z=4$

6) If $A=\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}, Using A^{-1} solve the system of equations

$2x-3y+5z=11$

$3x+2y-4z=-5$

$x+y-2z=-3.$

7) The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

8) The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is ₹ 70. Find cost of each item per kg by matrix method.

Answer

1) Given system of equations can be rewritten as $AX = B$

Where $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$

$$|A| = 2(10+3) - 1(-5-0) + 1(3-0)$$
$$= 26+5+3 = 34.$$

$$\operatorname{adj} A = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{|A|}(\operatorname{adj} A)B$$
$$= \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$$
$$= \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -3/2 \end{bmatrix}$$

$\therefore x = 1,\ y = \frac{1}{2},\ z = \frac{-3}{2}$

2) Try yourself $x=2,\ y=-1,\ z=1$

3) Try yourself $x=1,\ y=2,\ z=-1$

4) " $x=2,\ y=1,\ z=3$

5) " $x=1,\ y=2,\ z=3$

6) " $x=1,\ y=2,\ z=3.$

7) " $x=1,\ y=2,\ z=3$

8) " $x=5,\ y=8,\ z=8$

——x——x——