THREE DIMENSIONAL GEOMETRY.

One mark questions:

1) If a line makes angles 90^0, 135^0 and 45^0 with the x, y and z axes respectively. Find its direction cosines.

Solution:

Let $\alpha = 90^0$, $\beta = 135$, $\gamma = 45^0$

Let l, m, n are the direction cosines of a line

$\therefore l = \cos\alpha = \cos 90^0 = 0$

$m = \cos\beta = \cos 135^0 = -\frac{1}{\sqrt{2}}$,

$n = \cos\gamma = \cos 45^0 = \frac{1}{\sqrt{2}}$

2) If a line has direction ratio's -18, 12, -4. Then what are its direction cosines.

Solution:

x = - 18 y = 12 z = -4

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{(-18)^2+(12)^2+(-4)^2} = \sqrt{324+144+16} = \sqrt{484} = 22$$

Direction cosines are $l = \frac{x}{r} = \frac{-18}{22} = \frac{-9}{11}$

$$m = \frac{y}{r} = \frac{12}{22} = \frac{6}{11} \text{ and}$$

$$n = \frac{z}{r} = \frac{-4}{22} = \frac{-2}{11}$$

3) Find the direction cosines of x, y and z axis.

Solution:

The x – axis makes angles 0^0, 90^0, 90^0 with the positive direction of x, y and z – axis.

$\therefore$ Direction cosines of x – axis are $\cos 0^0$, $\cos 90^0$, $\cos 90^0$ i.e. 1, 0, 0.

Similarly direction cosines of y axis are $\cos 90^0$, $\cos 0^0$, $\cos 90^0$ i.e. 0, 1, 0

and direction cosines of z – axis are $\cos 90^0$, $\cos 90^0$, $\cos 0^0$, i.e. 0, 0, 1

4) Find the direction cosines of a line which makes equal angles with the co-ordinate axes.

Solution:

Let α, β, γ be the angles made by the line with the positive direction of x-axis, y –axis and z – axis

Also $\alpha = \beta = \gamma$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$3\cos^2\alpha = 1 \therefore \cos^2\alpha = \frac{1}{3} \quad \therefore \cos\alpha = \pm\frac{1}{\sqrt{3}}$

$\therefore$ The direction cosines are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$

5) Find the equation of the plane having intercept 3 on the y-axis and parallel to ZOX plane

Solution:

Y – intercept = b = 3

Any plane parallel to ZOX is y = b

The equation of the plane is y = 3

6) Find the distance of the plane 2x – 3y + 4z - 6 = 0 from the origin.

Solution:

Consider 2x – 3y + 4z – 6 = 0

2x – 3y + 4z = 6 – (1)

The Direction ratios are (2, -3, 4) = (x_1, y_1, z_1)

$$\therefore \quad r=\sqrt{2^2+(-3)^2+4^2}=\sqrt{4+9+16}=\sqrt{29}$$

The Direction cosines are $l=\frac{x_1}{r}=\frac{2}{\sqrt{29}}$

$$m=\frac{y_1}{r}=\frac{-3}{\sqrt{29}}$$

$$n=\frac{z_1}{r}=\frac{4}{\sqrt{29}}$$

Divide equation (1) by $\sqrt{29}$

$$\therefore \quad \frac{2}{\sqrt{29}}x-\frac{3}{\sqrt{29}}y+\frac{y}{\sqrt{29}}z=\frac{6}{\sqrt{29}}$$

and is of the form lx + my + nz = d

$\therefore$ The distance of the plane from origin is $=d=\frac{6}{\sqrt{29}}$

7) Find the equation of the plane which makes intercepts 1, -1 and 2 on the x, y and z axes respectively.

Solution:

a = x – intercept = 1, b = y – intercept = -1 and c = z – intercept = 2

The equation of the line is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ $\quad$ *i.e.* $\frac{x}{1}+\frac{y}{-1}+\frac{z}{2}=1$

8) Determine the direction cosines of the normal to the plane and the distance from the origin is x + y + z = 1

Solution:

Consider x + y + z = 1 - (1)

Direction ratio's of the plane are 1, 1, 1

$\therefore r=\sqrt{1^2+1^2+1^2}=\sqrt{1+1+1}=\sqrt{3}$ $\quad \therefore l=\frac{1}{\sqrt{3}}$ $\quad m=\frac{1}{\sqrt{3}}$ $\quad n=\frac{1}{\sqrt{3}}$

Divide equation (1) by $\sqrt{3}$ $\quad \therefore \frac{x}{\sqrt{3}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{3}}=\frac{1}{\sqrt{3}}$

It is of the form lx + my + nz = p

$\therefore$ P = distance from origin = $\frac{1}{\sqrt{3}}$

9) Find the intercepts cut off by the plane 2x + y – z = 5

Solution:

Consider 2x + y – z = 5

$$\frac{2x}{5}+\frac{y}{5}-\frac{z}{5}=1 \qquad i.e. \qquad \frac{x}{(5/2)}+\frac{y}{5}+\frac{z}{(-5)}=1$$

a = x– intercept = 5/2 b = y – intercept = 5 c = z – intercept = -5

10) Show that the planes 2x + y + 3z – 2 = 0 and x – 2y + 5 = 0 are perpendicular.

Solution:

Consider 2x + y + 3z - 2 = 0 i.e. ∴ 2x + y + 3z = 2

And x – 2y + 5 = 0 i.e. x – 2y + 0.z = - 5

The normals to the plane are

$$\overrightarrow{P}_1=2i+j+3k \quad \text{and} \quad \overrightarrow{P}_2=i-2j$$

$$\overrightarrow{P}_1.\overrightarrow{P}_2=2(1)+1(-2)+3(0)=2-2+0=0$$

∴ The planes $\overrightarrow{P}_1$ and $\overrightarrow{P}_2$ are perpendicular

11) Show that the planes 2x – y + 3z – 1 = 0 and 2x – y + 3z + 3 = 0 are parallel.

Solution:

Consider 2x – y + 3z – 1 = 0 i.e. 2x – y + 3z = 1

And 2x – y + 3z + 3 = 0 i.e. 2x – y + 3z = -3

∴ The normals to the plane are

$$\overrightarrow{P}_1=2i-j+3k \quad and\ \overrightarrow{P}_2=2i-j+3k$$

$$\therefore \quad \frac{a_1}{a_1}=\frac{2}{2}=1,\ \frac{b_1}{b_1}=\frac{-1}{-1}=1\ and\ \frac{c_1}{c_2}=\frac{3}{3}=1$$

$$\therefore \quad \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=1$$

∴ The planes P_1 and P_2 are parallel.

12) Find the equation of the plane parallel to x – axis and passing through the origin.

Solution:

The direction ratio's of x-axis is 1, 0, 0

The equation of the line through origin and parallel to x-axis

is $\frac{x-0}{1}=\frac{y-0}{0}=\frac{z-0}{0} \qquad i.e. \qquad \frac{x}{1}=\frac{y}{0}=\frac{z}{0}$

13) Find the vector equation of the straight line passing through (1.2.3) and perpendicular to the plane $\vec{r}.(i+2j-5k)+9=0$

Solution:

The required line passes through (1, 2, 3) and perpendicular to the plane

$\vec{r}.(i+2j-5k)+9=0$ is

$$\vec{r}=(i+2j+3k)+\lambda(i+2j-5k)$$

14) Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r}.(i+j+k)=2$

Solution:

Consider $\vec{r}.(i+j+k)=2 \quad \therefore x+y+z=2$

Any plane parallel to the given plane is x + y + z = λ

and is pass through (a, b, c) $\quad \therefore$ a + b + c = λ

Hence the equation of the plane parallel to the given plane is x + y + z = a + b + c

15) Find the distance between the two planes 2x+3y+4z=4 and 4x+6y+8z= 12.

Solution:

Consider 2x + 3y + 4z = 4 - (1)

And 4x + 6y + 8z = 12

i.e. 2x + 3y + 4z – 6 = 0 - (2)

$\therefore$ Distance from the point to the plane (2) = $\left|\frac{2x+3y+4z-6}{\sqrt{2^2+3^2+4^2}}\right|$

$$=\left|\frac{4-6}{\sqrt{4+9+16}}\right|=\left|\frac{-2}{\sqrt{29}}\right|=\frac{2}{\sqrt{29}}$$

Two mark questions:

1) Show that the points (2, 3, 4) (-1, -2, 1) and (5, 8, 7) are collinear.

Solution:

A = (2, 3, 4) B = (-1, -2, 1) and C = (5, 8, 7)

Direction ratio's of the line joining A & B are, 2+1, 3+2, 4-1, i.e. 3, 5, 3

Direction ratio's of the line joining B & C are -1-5, -2-8, 1-7, i.e. -6, -10, -6

$\therefore$ The direction ratio's of AB & BC are proportional & B is the common point of AB & BC

$\therefore$ The points A, B, C are collinear

2) Show that the line through the points (1, -1, 2) (3, 4, -2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6)

Solution:

Let A = (1, -1, 2) B = (3, 4, -2) C = (0, 3, 2) and D = (3, 5, 6)

Direction ratio's of AB are, a_1 = 3-1=2, b_1 = 4- (-1) = 4+1=5 & C_1 = -2-2 = -4

Direction ratio's of CD are a_2 = 3-0=3, b_2 = 5-3=2, C_2 = 6-2=4

Now $a_1a_2 + b_1b_2 + c_1c_2$ = 2 (3) + 5 (2) + (-4) 4

= 6+10 - 16 = 0

$\therefore$ AB is perpendicular to CD

3) Show that the line through the points (4, 7, 8) (2, 3, 4) is parallel to the line through the points (-1, -2, 1) (1, 2, 5).

Solution:

Let A = (4, 7, 8) B = (2, 3, 4) C = (-1, -2, 1) D = (1, 2, 5)

Direction ratio's of AB are $a_1 = 2 - 4 = -2$, $b_1 = 3-7=-4$, $c_1 = 4 - 8 = -4$

Direction ratio's of CD are $a_2 = 1- (-1) =1+1=2$, $b_2 = 2-(-2) = 2+2=4$, $c_2 = 5-1 =4$

$$\therefore \quad \frac{a_1}{a_2}=\frac{-2}{2}=-1, \quad \frac{b_1}{b_2}=\frac{-4}{4}=-1, \quad \frac{c_1}{c_2}=\frac{-4}{2}=-1$$

$\therefore \quad \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ Hence AB is parallel to CD

4) The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Write its equation in vector form.

Solution:

Consider $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$

$\therefore$ $\vec{a}$ = (5, -4, 6) and $\vec{b}$ = (3, 7, 2) are the direction ratio's

Vector equation of the line is $\overline{r}=\overline{a}+\lambda\overline{b}$

$$\overline{r}=\left(5i-4j+6k\right)+\lambda\left(3i+7j+2k\right)$$

5) Find the distance of the point (2, 3, -5) from the plane $\vec{r}\cdot(i+2j-2k)=9$

Solution:

Consider $\overline{r}.\left(i+2j-2k\right)=9 \ and \ \overline{a}=2i+3j-5k$

and $\overline{N}=i+2j-2k \quad and \ d=9$

$$\overline{a}.\overline{N}=2(1)+3(2)+(-5)(-2)=2+6+10=18$$

$$\left|\overline{N}\right|=\sqrt{1^2+2^2+(-2)^2}=\sqrt{1+4+4}=\sqrt{9}=3$$

Distance of a point from the plane = d = $\frac{\left|\overline{a}.\overline{N}-\overline{d}\right|}{\left|\overline{N}\right|}=\frac{18-9}{3}=\frac{9}{3}=3$

6) Find the equation of the plane passing through the line of intersection of the plane x + y + z = 6 and 2x + 3y + 4z – 5 = 0 and the point (1, 1, 1)

Solution:

Consider x + y + z = 6 $\quad \therefore$ x + y + z -6 = 0 and 2x + 3y + 4z – 5 = 0

The equation of the plane passing through the intersection of the two planes is x + y + z – 6 + λ (2x + 3y + 4z – 5) = 0 and is pass through (1, 1, 1)

$\therefore$ 1 + 1 + 1 – 6 + λ (2+3+4-5) = 0

- 3 + 4λ = 0 $\quad \therefore$ 4λ = 3 $\quad \therefore$ λ = ¾

The equation is (x + y + z – 6) + $\frac{3}{4}$ (2x + 3y + 4z – 5) = 0 (multiply by 4)

4x + 4y + 4z – 24 + 3 (2x + 3y + 4z – 5) = 0

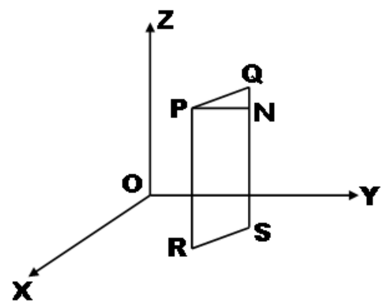
4x + 4y + 4z – 24 + 6x + 9y + 12z – 15 = 0

10x + 13y + 16z – 39 = 0

7) **Derive the direction cosine of a line passing through two points.**

Solution:

Let *l*, m, n be the direction cosines of a line PQ and the line PQ makes α, β and γ with positive directions of x, y and z axes respectively. Draw the perpendiculars from P and Q to xy – plane to meet at R & S and draw PN perpendicular to QS.



From the Δle PNQ, $P\hat{Q}N = \gamma$

$$\therefore \quad \cos\gamma = \frac{QN}{PQ} = \frac{ON - OQ}{PQ} = \frac{Z_2 - Z_1}{PQ}$$

$$Similarly \cos\alpha = \frac{x_2 - x_1}{PQ} \; and \quad \cos\beta = \frac{y_2 - y_1}{PQ}$$

Where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

8) **The Cartesian equation of a line is** $\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}$

Find the vector equation of the line

Solution:

Consider $\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}$

$$\therefore \quad \frac{x-(-3)}{2} = \frac{y-5}{4} = \frac{z-(-6)}{2}$$

$\therefore$ $x_1 = -3$ $y_1 = 5$ $z_1 = -6$ and a = 2 b = 4 and c = 2

$$\therefore \quad \vec{a} = (x_1, y_1, z_1) = (-3, 5, -6)$$

$\vec{b} = (a, b, c) = (2, 4, 2)$ are direction ratio's

$\therefore$ The vector equation of a line is $\vec{r} = \vec{a} + \lambda\vec{b}$

$$\vec{r} = (-3i + 5j - 6k) + \lambda(2i + 4j + 2k)$$

9) **Find the vector equation of the plane which is at a distance of 7 units from the origin and normal to the vector 3i + 5j – 6k**

Solution:

$$let \; \vec{n} = 3i + 5j - 6k \qquad and \; |\vec{n}| = \sqrt{3^2 + 5^2 + (-6)^2} = \sqrt{9 + 25 + 36} = \sqrt{70}$$

$$and \; \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3i + 5j - 6k}{\sqrt{70}} = \left(\frac{3}{\sqrt{70}}i + \frac{5}{\sqrt{70}}j - \frac{6}{\sqrt{70}}k\right)$$

$\therefore$ The equation of the plane $\vec{r}.\hat{n} = \vec{d} \; and \; d = 7$

$$\therefore \quad \vec{r}.\left(\frac{3}{\sqrt{70}}i + \frac{5}{\sqrt{70}}j - \frac{6}{\sqrt{70}}k\right) = 7$$

10) Find the distance of the point (3, -2, 1) from the plane 2x – y + 2z + 3 = 0

Solution:

Consider 2x – y + 2z + 3 = 0

$$\therefore \quad d=\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|=\left|\frac{2(3)+(-1)(-2)+2(1)+3}{\sqrt{2^2+(-1)^2+2^2}}\right|$$

$$=\left|\frac{6+2+2+3}{\sqrt{4+1+4}}\right|=\frac{13}{3}$$

<u>**Three mark questions**</u>

1) Find the vector and Cartesian equations of the line that passes through the points (3, -2, -5) and (3, -2, 6)

Solution:

Let A = (3, -2, -5)   B = (3, -2, 6)

Direction ratio's of AB are, a = 3 – 3 = 0

b = -2 – (-2) = - 2+2 = 0

c = 6 – (-5) = 6 + 5 = 11

$\therefore\ \vec{b}=ai+bj+ck=0.i+0.j+11k=11k\ \ and\ \ \vec{a}=(3,-2,-5)$ = 3i – 2j – 5k

$\therefore$ Vector equation of a line passing through two points is $\vec{r}=\vec{a}+\lambda\,\vec{b}$

$\vec{r}=3i-2j-5k+\lambda(11k)$

Cartesian equation of a line is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

$$\frac{x-3}{0}=\frac{y-(-2)}{0}=\frac{z-(-5)}{11}\ \ i.e.\ \ \frac{x-3}{0}=\frac{y+2}{0}=\frac{z+5}{11}$$

2) Show that three lines with direction cosine $\frac{12}{13},\frac{-3}{13},\frac{-4}{13};\ \frac{4}{13},\frac{12}{13},\frac{3}{13};\ \frac{3}{13},\frac{-4}{13},\frac{12}{13}$ are mutually perpendicular.

Solution:

L_1, L_2, L_3 are three lines.

The direction cosine of the line $L_1=\left(\frac{12}{13},\frac{-3}{13},\frac{-4}{13}\right)=(l_1,m_1,n_1)$

Direction cosines of the line $L_2=\left(\frac{4}{13},\frac{12}{13},\frac{3}{13}\right)=(l_2,m_2,n_2)$

Direction cosines of the line $L_3=\left(\frac{3}{13},\frac{-4}{13},\frac{12}{13}\right)=(l_3,m_3,n_3)$

$$\therefore\ l_1l_2+m_1m_2+n_1n_2=\frac{12}{13}\left(\frac{4}{13}\right)+\left(\frac{-3}{13}\right)\left(\frac{12}{13}\right)+\left(\frac{-4}{13}\right)\left(\frac{3}{13}\right)=\frac{48-36-12}{169}=0$$

$\therefore \quad L_1\ is\ perpendicular\ to\ L_2$

$$\therefore\ l_2l_3+m_2m_3+n_2n_3=\frac{4}{13}\cdot\left(\frac{3}{13}\right)+\frac{12}{13}\left(\frac{-4}{13}\right)+\frac{3}{13}\cdot\left(\frac{12}{13}\right)=\frac{12-48+36}{169}=\frac{48-48}{169}=0$$

$\therefore \quad L_2\ is\ perpendicular\ to\ L_3$

$\therefore \; l_3 l_1 + m_3 m_1 + n_3 n_1 = \frac{3}{13}\cdot\left(\frac{12}{13}\right) + \left(\frac{-4}{13}\right)\left(\frac{-3}{13}\right) + \frac{12}{13}\left(\frac{-4}{13}\right) = \frac{36+12-48}{169} = \frac{48-48}{169} = 0$

$\therefore \quad L_3 \text{ is perpendicular to } L_1$

Hence the three lines are mutually perpendicular

3) Find the angle between the pair of lines $\vec{r} = 3i + 5j - k + \lambda(i + j + k)$ **and** $\vec{r} = 7i + 4k + \mu(2i + 2j + 2k)$

Solution:

Consider $\vec{r} = 3i + 5j - k + \lambda(i + j + k) \quad \therefore \quad \vec{b_1} = i + j + k$

$\vec{r} = 7i + 4k + \mu(2i + 2j + 2k) \quad \therefore \quad \vec{b_2} = 2i + 2j + 2k$

$\therefore \quad \vec{b_1}.\vec{b_2} = 1(2) + 1(2) + 1(2) = 2 + 2 + 2 = 6$

$\left|\vec{b_1}\right| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \qquad \left|\vec{b_2}\right| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$

$\therefore \quad \cos\theta = \frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|} = \frac{6}{2\sqrt{3}.\sqrt{3}} = \frac{6}{2\times 3} = \frac{6}{6} = 1 = \cos 0^0$

$\therefore \quad \theta = 0^0$

4) Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector 3i + 2j – 2k, both in vector form and Cartesian form.

Solution:

Let $\vec{a} = (1, 2, 3) = i + 2j + 3k \text{ and } \vec{b} = 3i + 2j - 2k$

The vector equation of the line is $\vec{r} = \vec{a} + \lambda\,\vec{b}$

$\therefore \quad \vec{r} = (i + 2j + 3k) + \lambda(3i + 2j - 2k)$

Let $\vec{r}$ be the position vector of the point and $\vec{r} = xi + yj + zk$

$\therefore \quad xi + yj + zk = (i + 2j + 3k) + \lambda(3i + 2j - 2k)$

$= i + 2j + 3k + 3\lambda\, i + 2\lambda j - 2\lambda k$

$= (1 + 3\lambda)i + (2 + 2\lambda)\, j + (3 - 2\lambda)k$

$\therefore \quad x = 1 + 3\lambda \qquad 2 + 2\lambda = y \quad \text{and} \quad z = 3 - 2\lambda$

$x - 1 = 3\lambda \qquad 2\lambda = y - 2 \qquad z - 3 = -2\lambda$

$\frac{x-1}{3} = \lambda \qquad \lambda = \frac{y-2}{2} \quad \therefore \quad \lambda = \frac{z-3}{-2}$

$\therefore \quad \frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{-2}$ is the equation of the line in Cartesian form.

5) Find the distance between parallel lines $\vec{r}=i+2j-4k+\lambda(2i+3j+6k)$ **and** $\vec{r}=3i+3j-5k+\mu(2i+3j+6k)$

Solution:

Consider $\vec{r}=i+2j-4k+\lambda(2i+3j+6k)$

And $\vec{r}=3i+3j-5k+\mu(2i+3j+6k)$

$\therefore \quad \vec{a}_1=i+2j-4k \qquad \vec{b}_1=2i+3j+6k$

and $\quad \vec{a}_2=3i+3j-5k \qquad \vec{b}_2=2i+3j+6k$

$\therefore \quad \vec{b}_1=\vec{b}_2$ $\therefore$ The lines are parallel

$\therefore \quad \vec{b}=\vec{b}_1=\vec{b}_2=2i+3j+6k \qquad and\ |\vec{b}|=\sqrt{2^2+3^2+6^2}$

$\therefore \quad \vec{a}_2-\vec{a}_1=2i+j+10k \qquad =\sqrt{4+9+36}=\sqrt{49}=7$

$$\therefore \quad \vec{b}\times(\vec{a}_2-\vec{a}_1)=\begin{vmatrix} i & j & k \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}=i(-3-6)-j(-2-12)+k(2-6)=-9i+14j-4k$$

$$\left|\vec{b}\times(\vec{a}_2-\vec{a}_1)\right|=\sqrt{(-9)^2+(4)^2+(-4)^2}=\sqrt{81+196+16}=\sqrt{293}$$

$\therefore$ Distance between parallel lines = d = $\left|\dfrac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|=\dfrac{\sqrt{293}}{7}$

6) Find the angle between the pair of lines $\dfrac{x+3}{3}=\dfrac{y-1}{5}=\dfrac{z+3}{4}$ $and\ \dfrac{x+1}{1}=\dfrac{y-4}{1}=\dfrac{z-5}{2}$

Solution:

Consider $\dfrac{x+3}{3}=\dfrac{y-1}{5}=\dfrac{z+3}{4} \quad -(1) \quad \therefore$ *Direction ratios of* $\vec{b}_1=(3,5,4)$

and $\dfrac{x+1}{1}=\dfrac{y-4}{1}=\dfrac{z-5}{2} \quad -(2) \quad$ *Direction ratio's of* $\vec{b}_2=(1,1,2)$

$\vec{b}_1.\vec{b}_2=3(1)+5(1)+4(2)=3+5+8=16$

$|\vec{b}_1|=\sqrt{3^2+5^2+4^2}=\sqrt{9+25+16}=\sqrt{50}=\sqrt{25\times 2}=5\sqrt{2}$

$|\vec{b}_2|=\sqrt{1^2+1^2+2^2}=\sqrt{1+1+4}=\sqrt{6}$

$$\therefore \quad \cos\theta=\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}=\frac{16}{5\sqrt{2}\sqrt{6}}=\frac{16}{5\sqrt{12}}=\frac{16}{5\sqrt{4\times 3}}=\frac{16}{5\times 2\sqrt{3}}=\frac{8}{5\sqrt{3}}$$

$$\therefore \quad \theta=\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

7) **Find the shortest distance between the lines**

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \quad and \quad \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

Solution:

Consider $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ *and* $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

i.e. $\frac{x-(-1)}{7}=\frac{y-(-1)}{-6}=\frac{z-(-1)}{1}$ $\quad \therefore \quad \vec{a}_2 = 3i+5j+7k$

$\therefore \quad \vec{a}_1 = -i-j-k \qquad \vec{b}_2 = i-2j+k$

$\vec{b}_1 = 7i-6j+k$

$\vec{a}_2-\vec{a}_1 = 3i+5j+7k+i+j+k = 4i+6j+8k$

$$\vec{b}_1\times\vec{b}_2 = \begin{vmatrix} i & j & k \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} = i(-6+2)-j(7-1)+k(-14+6)$$

$$= -4i-6j-8k$$

$$\left|\vec{b}_1\times\vec{b}_2\right| = \sqrt{(-4)^2+(-6)^2+(-8)^2} = \sqrt{16+36+64} = \sqrt{116}$$

$\therefore$ Shortest distance = $d = \left|\frac{(\vec{b}_1\times\vec{b}_2).(\vec{a}_2-\vec{a}_1)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| = \left|\frac{-16-36-64}{\sqrt{116}}\right|$

$$= \left|\frac{-116}{\sqrt{116}}\right| = \sqrt{116} = \sqrt{4\times 29} = 2\sqrt{29}$$

8) **Find the equation of the planes passing through three points (1, 1, 0) (1, 2, 1) and (-2, 2, -1)**

Solution:

Let $\vec{a}$ = (1, 1, 0) $\quad \vec{b}$ = (1, 2, 1) and $\vec{c}$ = (-2, 2, -1) and $\vec{r} = xi+yj+zk$

$\vec{r}-\vec{a} = (x-1)i+(y-1)j+(z-0)k$

$\overrightarrow{AB} = \vec{b}-\vec{a} = (0, 1, 1)$ *and* $\quad \overrightarrow{AC} = \vec{c}-\vec{a} = (-3, 1, -1)$

The vector equation of the plane is $(\vec{r}-\vec{a}).(\overrightarrow{AB}\times\overrightarrow{AC}) = 0$

$$\begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

(x-1) (-1-1) – (y-1) (0+3) + z (0 + 3) = 0

-2(x-1) -3 (y-1) + 3z = 0

-2x + 2 – 3y + 3 + 3z = 0

-2x – 3y + 3z + 5 = 0

2x + 3y - 3z - 5 = 0

$\therefore$ 2x + 3y – 3z = 5 is the equation of the plane

9) **Find the angle between the pair of lines given by** $\vec{r} = 3i+2j-4k+\lambda(i+2j+2k)$ **and** $\vec{r} = 5i-2j+\mu(3i+2j+6k)$**.**

Solution: $\quad \vec{b}_1 = i+2j+2k \; \vec{b}_2 = 3i+2j+6k$

$$\therefore \cos\theta = \left|\frac{\vec{b}_1.\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right| = \left|\frac{3+4+12}{\sqrt{9}\sqrt{49}}\right| = \frac{19}{21} \quad \therefore \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

10) Prove that if a plane has intercepts a, b, c and is at a distance of p units from the origin then $\frac{1}{a_2}+\frac{1}{b_2}+\frac{1}{c_2}=\frac{1}{p^2}$

Solution:

Let a, b, c, are the intercepts of the plane

And the equation is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad -(1)$

$\therefore$ P = The distance of the plane (1) from (0, 0, 0)

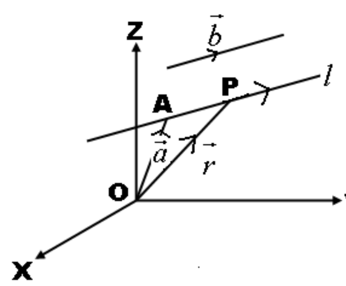
$$P=\left|\frac{0+0+0-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=\left|\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|$$

$$P^2=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} \qquad \therefore \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Five mark questions:

1) Derive the equation of the line in space passing through a point and parallel to a vector, both in the vector form and Cartesian form.

Solution:



Let $\vec{a}$ be the position vector of the given point A. w.r. to the origin O of the rectangular co-ordinate system. Let *l* be the line which passes through the point A and is parallel to the given vector $\vec{b}$. Let $\vec{r}$ be the position vector of an arbitrary point P on the line. Then $\overrightarrow{AP}$ is parallel to $\vec{b}$.

i.e. $\overrightarrow{AP}=\lambda\,\vec{b}$ where λ is a real number

$\overrightarrow{OP}-\overrightarrow{OA}=\lambda\,\vec{b}$

$\vec{r}-\vec{a}=\lambda\,\vec{b}$

$\therefore\ \vec{r}=\vec{a}+\lambda\vec{b}$ *is the vector equation of the line*

Let A = (x_1, y_1, z_1) be the co-ordinates of the given point and the direction ratio's of the line are a, b, c.

Let P = (x, y, z) be the co-ordinate of any point

Then $\vec{r}=xi+yj+zk$ *and* $\vec{a}=x_1i+y_1j+z_1k$ *and* $\vec{b}=ai+bj+ck$ *and* $\vec{r}=\vec{a}+\lambda\vec{b}$

$xi + yj + zk = (x_1i + y_1j + z_1k) + \lambda (ai + bj + ck)$

$= x_1i + y_1j + z_1k + \lambda ai + \lambda bj + \lambda ck$

$= (x_1 + \lambda a) i + (y_1 + \lambda b) j + (z_1 + \lambda c) k$

Equating the coefficients of i, j and k we get

$x = x_1 + \lambda a \qquad y = y_1 + \lambda b \qquad$ and $z = z_1 + \lambda c$

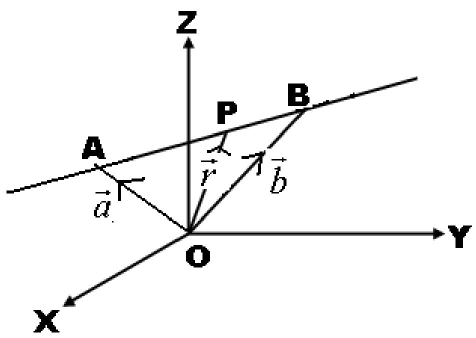
these are the parametric equations of a line

$\therefore\ x - x_1 = \lambda a \qquad y - y_1 = \lambda b \qquad$ and $z - z_1 = \lambda c$

$\therefore \qquad \frac{x-x_1}{a}=\lambda \qquad \frac{y-y_1}{b}=\lambda \qquad \frac{z-z_1}{c}=\lambda$

$\therefore \qquad \frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$. This is the Cartesian equation of the line.

2) **Derive the equation of a line in space passing through two given points both invector form and Cartesian form.**
Solution:



Let $\vec{a}$ & $\vec{b}$ & $\vec{r}$ are the position vectors of the two points A (x_1, y_1, z_1) is (x_2, y_2, z_2) and p (x, y, z) respectively.

$\vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - \vec{a}$ and $\vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$

If the point p lien on the line $\vec{AB}$ if and only if $\vec{AP}$ and $\vec{AB}$ are collinear.

$\therefore \quad \vec{AP} = \lambda \ \vec{AB} \qquad i.e. \quad \vec{r} - \vec{a} - \lambda(\vec{b} - \vec{a})$

$\therefore \quad \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ is the vector equation of the line passing through two points.

Let $\vec{r} = xi + yj + zk, \quad \vec{a} = x_1 i + y_1 j + z_1 k \quad \vec{b} = x_2 i + y_2 j + z_2 k \quad \& \ \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

$$xi + yj + 2k = x1i + y1j + z1k + \lambda\left[(x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k\right]$$
$$= \left[x_1 + \lambda(x_2 - x_1)\right]i + \left[y_1 + \lambda(y_2 - y_1)\right]j + \left[z_1 + \lambda(z_2 - z_1)\right]k$$

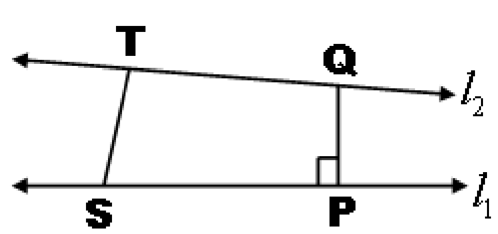
$\therefore \quad x = x_1 + \lambda(x_2 - x_1), \qquad y = y_1 + \lambda(y_2 - y_1) \quad \& \quad z = z_1 + \lambda(z_2 - z_1)$

$x - x_1 = \lambda(x_2 - x_1) \qquad y - y_1 = \lambda(y_2 - y_1) \qquad z - z_1 = \lambda(z_2 - z_1)$

$\dfrac{x - x_1}{x_2 - x_1} = \lambda \qquad \dfrac{y - y_1}{y_2 - y_1} = \lambda \qquad \therefore \quad \dfrac{z - z_1}{z_2 - z_1} = \lambda$

$\therefore \quad \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$ is the Cartesian equation of the line passing through two points.

3) **Derive the shortest distance between two skew lines both in vector form and Cartesian form.**
Proof:



Let l_1 and l_2 be the skew lines

Let $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ be the skew lines. Let s and T are any two points on l_1 and l_2 with position vectors $\vec{a}_1$ and $\vec{a}_2$ respectively.

Then the magnitude of the shortest distance is equal to the projection of ST along the direction of a line.

If $\vec{PQ}$ is the shortest distance between the lines l_1 and l_2 then it is perpendicular to both $\vec{b}_1$ and $\vec{b}_2$ and $\hat{n}$ is the unit vector along $\vec{PQ}$.

$\therefore \qquad \hat{n} = \dfrac{\vec{b}_1 \times \vec{b}_2}{\left|\vec{b}_1 \times \vec{b}_2\right|}$ let θ be the angle between $\overrightarrow{ST}$ *and* $\overrightarrow{PQ}$

Then $PQ = ST\ \cos\theta$ *and* $\cos\theta = \left|\dfrac{\overrightarrow{PQ}.\overrightarrow{ST}}{\left|\overrightarrow{PQ}\right|\left|\overrightarrow{ST}\right|}\right|$ $\qquad$ $but\left|\overrightarrow{PQ}\right| = d\ \ and\ \ ST = a_2 - a_1$

$$\cos\theta = \left|\frac{d\,\hat{n}\left(\vec{a}_2 - \vec{a}_1\right)}{d\ .\ ST}\right|$$

$$\therefore \qquad ST\ \cos\theta = \left|\frac{\left(\vec{b}_1 \times \vec{b}_2\right)\left(\vec{a}_2 \times \vec{a}_1\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|}\right|$$

Shortest distance is d = PQ = $|ST||\cos\theta| = \left|\dfrac{\left(\vec{b}_1 \times \vec{b}_2\right)\left(\vec{a}_2 \times \vec{a}_1\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|}\right|$

is the Shortest distance of skew lines in vector form.

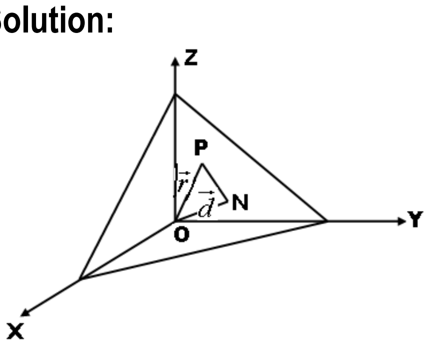
Let $l_1 : \dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $l_2 = \dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$ be the equations of two skew lines in Cartesian form.

The shortest distance between two skew lines is

$d = \left|\dfrac{\Delta}{\sqrt{\left(b_1c_2 - b_2c_1\right)^2 + \left(c_1a_2 - c_2a_1\right)^2 + \left(a_1b_2 + a_2b_1\right)^2}}\right|$ where $\Delta = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

4) Derive the equation of the plane in normal form both in the vector form and Cartesian form.

Solution:



Consider a plane whose perpendicular distance from the origin is d. If $\overrightarrow{ON}$ is the normal from the origin to the plane and $\hat{n}$ is the unit normal vector $\overrightarrow{ON}$

Then $\overrightarrow{ON} = d.\hat{n}$

Let P be any point on the plane then $\overrightarrow{NP}$ is perpendicular to $\overrightarrow{ON}$

$\therefore \qquad \overrightarrow{NP}.\overrightarrow{ON} = 0 \qquad\qquad -(1)$

Let $\vec{r}$ be the position vector of the point P

Then $\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON} = \vec{r} - d.\hat{n}$

From equation (1) $\left(\vec{r} - d.\hat{n}\right).d.\hat{n} = 0 \qquad But\ d \neq 0$

$\therefore \left(\vec{r} - d.\hat{n}\right)\hat{n} = 0$

$\therefore \vec{r}.\hat{n} - d.\hat{n}.\hat{n} = 0 \qquad But\ \hat{n}.\hat{n} = 1.1 = 1$

$\therefore \vec{r}.\hat{n} - d = 0$

$\therefore \vec{r}.\hat{n} = d$ is the equation of the plane vector form

Let *l*, m, n be the direction cosines of $\hat{n}$

Then $\hat{n} = li + mj + nk \text{ and } \overrightarrow{OP} = \vec{r} = xi + yj + zk$

$\therefore \vec{r}.\hat{n} = d$

$(xi + yj + zk).(li + mj + nk) = d$

Therefore lx + my + nz = d is the Cartesian equation of the plane in normal form

5) **Derive the condition for the coplanarity of two lines in space both in the vector form and Cartesian form.**

Solution:

Let the given lines be $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \text{ and } \vec{r} = \vec{a}_2 + \lambda \vec{b}_2 \qquad -(2)$

The line (1) passes through the point A with the position vector $\vec{a}_1$ and parallel to $\vec{b}_1$ and the line (2) passes through the point B with the position vector $\vec{a}_2$ and parallel to $\vec{b}_2$

Thus $\overrightarrow{AB} = \vec{B} - \vec{A} = \vec{a}_2 - \vec{a}_1$

The given lines are coplanar if and only if $\overrightarrow{AB}$ is perpendicular to $\vec{b}_1 \times \vec{b}_2$

i.e. $\overrightarrow{AB}.(\vec{b}_1 \times \vec{b}_2) = 0$

$(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$ is condition for the coplanarity of two lines in vector form.

Let A = (x_1, y_1, z_1) and B = (x_2, y_2, z_2) be the co-ordinates of the points A and B respectively. Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratio's of $\vec{b}_1 \text{ and } \vec{b}_2$ respectively.

Then $\overrightarrow{AB} = B - A = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

$\vec{b}_1 = a_1 i + b_1 j + c_1 k \qquad and \qquad \vec{b}_2 = a_2 i + b_2 j + c_2 k$

$\therefore$ The given lines are coplanar if $\overrightarrow{AB}.(\vec{b}_1 \times \vec{b}_2) = 0$

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

is condition for coplanarity of two lines in cartestion form.