MATHEMATICS
II PUC
VECTOR ALGEBRA
QUESTIONS & ANSWER

I One Mark Question

1) Find the unit vector in the direction of $2i + 3j + k$.

Let $\vec{a} = 2i + 3j + k$

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2i + 3j + k}{\sqrt{14}}$$

$$\therefore \hat{a} = \frac{2}{\sqrt{14}} i + \frac{3}{\sqrt{14}} j + \frac{1}{\sqrt{14}} k$$

2) Let $\vec{a} = i + 2j$ & $\vec{b} = 2i + j$. If $|\vec{a}| = |\vec{b}|$. Are the vectors $\vec{a}$ & $\vec{b}$ equal?

$$|\vec{a}| = \sqrt{a^2 + 2^2} = \sqrt{5}, \qquad |\vec{b}| = \sqrt{2^2 + a^2} = \sqrt{5}$$

$$\therefore |\vec{a}| = |\vec{b}|$$

But vectors are not equal since the corresponding components are distinct i.e. directions are different.

3) Find the values of x & y so that vectors 2i+3j and xi+4j are equal.

$$\vec{a} = 2i + 3j \qquad \vec{b} = xi + yj$$

$$Given\ \vec{a} = \vec{b} \qquad \therefore 2i + 3j = xi + yi$$

$$\therefore x = 2, \ y = 3$$

4) Find the scalar or dot product of vectors $i + 2j - 3k$ & $2i - j + k$.

$$(i + 2j - 3k) \ . \ (2i - j + k) = 1(2) + 2(-1) - 3(1) = 2 - 2 - 3 = -3$$

5) Show that $\frac{i - j}{\sqrt{2}}$ is a unit vector.

$$\vec{a} = \frac{i - j}{\sqrt{2}} = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$$

$$|\vec{a}| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1}$$

$\therefore |\vec{a}| = 1 \qquad \therefore \vec{a}$ is a unit vector.

II **Two Marks Questions:**

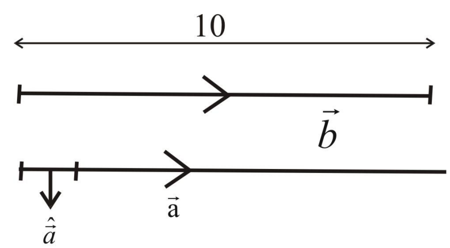
1) Find the vector parallel to the vector i - 2j and has magnitude 10 units.

$$\text{Let } \vec{a} = i - 2j$$

$$|\vec{a}| = \sqrt{1^2 + (-2)^2}$$

$$\therefore |\vec{a}| = \sqrt{5}$$

$$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{i - 2j}{\sqrt{5}}$$



Let $\vec{b}$ be the vectors parallel to $\vec{a}$ having magnitude 10 units

$\therefore |\vec{b}| = 10$

∵ If vectors are parallel then unit vector along & parallel vectors are same

Now $\vec{b} = |\vec{b}| \, \hat{a}$

$$= 10 \cdot \frac{i - 2j}{\sqrt{5}}$$

$$\therefore \text{Reqd vector } \vec{b} = \frac{10i}{\sqrt{5}} - \frac{20j}{\sqrt{5}}$$

2) Find the direction ratios and direction cosines of the vector $\vec{a} = i + j - 2k$.

$$\vec{a} = i + j - 2k$$

$$|\vec{a}| = \sqrt{1^2 + 1^2 + (-2)^2}$$

$$\therefore |\vec{a}| = \sqrt{6}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{i + j - 2k}{\sqrt{6}}$$

$$\therefore \hat{a} = \frac{1}{\sqrt{6}} i + \frac{1}{\sqrt{6}} j - \frac{2}{\sqrt{6}} k$$

Here direction ratios are components of $\vec{a}$ i.e. $(1, 1, -2)$

direction cosines are components of $\hat{a}$ i.e. $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

3) Show the vectors 2i-3j+4k and -4i+6j-8k are collinear.

$\vec{a} = 2i - 3j + 4k$

$\vec{b} = -4i + 6j - 8k = -2(2i - 3j + 4k) = -2\vec{a}$

$\therefore$ One vector can be expressed interms of another

$\therefore \vec{a}$ & $\vec{b}$ are collinear.

4) Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$

Given $|\vec{a}| = 1$

$(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$

$|\vec{x}|^2 - |\vec{a}|^2 = 12$

$|\vec{x}|^2 - 1^2 = 12$

$|\vec{x}|^2 = 13$

$\therefore |\vec{x}| = \sqrt{13}$

5) Find $|\vec{a} - \vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2, |\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 4$

Given $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 4$

$w.k.t.\ |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 - 2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2$

$= 2^2 - 2(4) + 3^2 = 4 - 8 + 9$

$|\vec{a} - \vec{b}|^2 = 5$

$\therefore |\vec{a} - \vec{b}| = \sqrt{5}$

6) For any two vectors $\vec{a}$ and $\vec{b}$ prove that $|\vec{a} \bullet \vec{b}| \le |\vec{a}| \bullet |\vec{b}|$

$$w.k.t.\ \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|\ |\vec{b}|}$$

$$\frac{|\vec{a}.\ \vec{b}|}{|\vec{a}|\ |\vec{b}|} = |\cos\theta| \qquad \text{for all values of } \theta,\ -1 \le \cos\theta \le 1$$

$$\therefore |\cos\theta| \le 1$$

$$\therefore \frac{|\vec{a}.\ \vec{b}|}{|\vec{a}|\ |\vec{b}|} \le 1$$

$$\therefore |\vec{a}.\ \vec{b}| \le |\vec{a}|\ |\vec{b}|$$

7) For any two vectors $\vec{a}$ and $\vec{b}$ prove that $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$ (Triangle in equality)

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2(\vec{a}.\ \vec{b}) + |\vec{b}|^2$$

$$\le |\vec{a}|^2 + 2|\vec{a}.\ \vec{b}| + |\vec{b}|^2 \qquad \because \vec{a}.\vec{b} \le |\vec{a}.\vec{b}|$$

$$\le |\vec{a}|^2 + 2|\vec{a}|\ |\vec{b}| + |\vec{b}|^2 \qquad \text{From previous properties } |\vec{a}.\vec{b}| \le |\vec{a}||\vec{b}|$$

$$\le \{|\vec{a}| + |\vec{b}|\}^2$$

$$\therefore |\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$$

8) Evaluate $(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b})$

$$(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b}) = 6\vec{a}.\vec{a} + 21(\vec{a}.\vec{b}) - 10(\vec{b}.\vec{a}) - 35(\vec{b}.\vec{b})$$

$$= 6\vec{a}^2 + 21(\vec{a}.\vec{b}) - 10(\vec{a}.\vec{b}) - 35\vec{b}^2$$

$$= 6\vec{a}^2 + 11\vec{a}.\vec{b} - 35\vec{b}^2$$

$$= 6|\vec{a}|^2 + 11\vec{a}.\vec{b} - 35\vec{b}^2$$

9) If $\vec{a} = i - 7j + 7k$ & $\vec{b} = 3i - 2j + 2k$ find $\vec{a} \times \vec{b}$ and $|\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= i\{-7(2) - (-2)(7)\} - j\{2 - 21\} + k\{-2 + 21\}$$

$$= i(-14 + 14) - j\{-19\} + k(19)$$

$$\therefore \vec{a} \times \vec{b} = 19j + 19k$$

10) Find λ & μ if $(2i+6j+27k)\times(i+\lambda j+\mu k)=\vec{o}$

Given $(2i+6j+27k)\times(i+\lambda j+\mu k)=\vec{o}$

$$\therefore \begin{vmatrix} i & j & k \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = \vec{o}$$

$i(6\mu-27\lambda)-j(2\mu-27)+k(2\lambda-6)=\vec{o}$

Equating cofficients

$6\mu - 27\lambda = 0, \qquad 2\mu - 27 = 0 \qquad 2\lambda - 6 = 0$

$$\therefore \mu = \frac{27}{2} \text{ and } \lambda = 3$$

11) Show that $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$

Consider LHS $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$

$=\vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{b}\times\vec{b}$

$=\vec{o}+(\vec{a}\times\vec{b})+(\vec{a}\times\vec{b})-\vec{o}$

$\therefore \quad (\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$

12) Find the scalar triple product of vectors $i+2j+3k$, $-i-j+k$ and $i+j+k$

$$\text{Scalar triple product} = \begin{vmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$=1(-1-1)-2(-1-1)+3(-1+1)$

$=-2+4+0=2$

13) Find λ if the vectors $i+j+2k$, $\lambda i-j+k$ & $3i-2j-k$ are coplanar.

Given that vectors are coplanar

$$\therefore \begin{vmatrix} 1 & 1 & 2 \\ \lambda & -1 & 1 \\ 3 & -2 & -1 \end{vmatrix} = 0$$

$=1(1+2)-1(-\lambda-3)+2(-2\lambda+3)=0$

$=3+\lambda+3-4\lambda+6=0$

$=-3\lambda+12=0$

$\therefore \lambda = 4$

III **Three Marks Questions:**

1) Consider the points P and Q with position vectors $\overrightarrow{OP} = 3\vec{a} - 2\vec{b}$ and $\overrightarrow{OQ} = \vec{a} + \vec{b}$. Find the position vector of a point R which divides line joining the points P and Q in the ratio $2:1$ internally and externally respectively.

Solution

given $\overrightarrow{OP} = 3\vec{a} - 2\vec{b}$, $\overrightarrow{OQ} = \vec{a} + \vec{b}$ $\quad m:n = 2:1$

Internally,

$$\overrightarrow{OR} = \frac{m\overrightarrow{OQ} + n\overrightarrow{OP}}{m+n}$$

$$= \frac{2(\vec{a} + \vec{b}) + 1(3\vec{a} - 2\vec{b})}{2+1}$$

$$= \frac{2\vec{a} + 3\vec{a} + 2\vec{b} - 2\vec{b}}{2+1}$$

$$\overrightarrow{OR} = \frac{5\vec{a}}{3}$$

externally $\overrightarrow{OR} = \dfrac{m\overrightarrow{OQ} - n\overrightarrow{OP}}{m-n}$

$$= \frac{2(\vec{a} + \vec{b}) - 1(3\vec{a} - 2\vec{b})}{2-1}$$

$$= \frac{2\vec{a} - 3\vec{a} + 2\vec{b} + 2\vec{b}}{1}$$

$$\overrightarrow{OR} = 4\vec{b} - \vec{a}$$

3) Find the vector joining the points P(2, 3, 0) and Q(-1, -2, 4) and also direction cosines of $\overrightarrow{PQ}$

Given $P \equiv (2,3,0)$ $\quad Q \equiv (-1,-2,\ 4)$

Let i, j, k be unit vectors along axes

then $\overrightarrow{OP} = 2i + 3j$

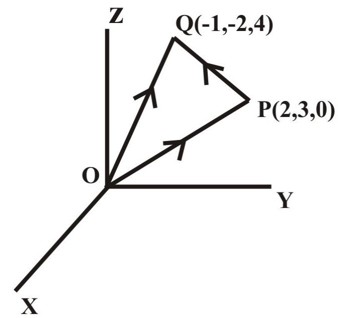
$\overrightarrow{OQ} = -i - 2j + 4k$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= -i - 2j + 4k - (2i + 3j)$

$= - i - 2i + 4j - 2i - 3j$

$\therefore \overrightarrow{PQ} = -3i - 5j + 4k$



$\left|\overrightarrow{PQ}\right| = \sqrt{(-3)^2 + (-5)^2 + (4)^2} = \sqrt{9 + 25 + 16} = \sqrt{50}$

$\therefore \left|\overrightarrow{PQ}\right| = 5\sqrt{2}$

$\overrightarrow{\hat{PQ}} = \dfrac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \dfrac{-3i - 5j + 4k}{5\sqrt{2}}$

$\therefore \overrightarrow{\hat{PQ}} = \dfrac{-3}{5\sqrt{2}} i - \dfrac{5}{5\sqrt{2}} j + \dfrac{4}{5\sqrt{2}} k$

$\therefore$ direction cosines are $\left(\dfrac{-3}{5\sqrt{2}},\ \dfrac{1}{\sqrt{2}},\ \dfrac{4}{5\sqrt{2}} \right)$

4) Show that points A(1, 2, 7) , B(2, 6,3) & C(3, 10, -1) are collinear OR

Show that the points with position vectors i+2j+7k, 2i+6j+3k and 3i+10j-k are collinear.

$\overrightarrow{OA} = (1,2,7)$ $\quad \overrightarrow{OB} = (2,6,3)$ $\quad \overrightarrow{OC} = (3,10,\ -1)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2-1, 6-2, 3-7) = (1,4,-4)$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (3,10,-1) - (2,6,3) = (1,4,-4)$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3,10,\ -1) - (1,2,7) = (2,8,-8)$

$\left|\overrightarrow{AB}\right| = \sqrt{33}$ $\quad \left|\overrightarrow{BC}\right| = \sqrt{33}$ $\quad \left|\overrightarrow{AC}\right| = \sqrt{132}$

$\therefore \left|\overrightarrow{AC}\right| = 2\sqrt{33}$

Here $\left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right| = \left|\overrightarrow{AC}\right|$

$\therefore$ Collinear condition is satisfied.

$\therefore$ A, B, C are collinear.

5) Find the angle between the vectors $i - 2j + 3k$ & $3i - 2j + k$.

Let $\vec{a} = i - 2j + 3k$ $\vec{b} = 3i - 2j + k$

$\vec{a} . \vec{b} = (i - 2j + 3k) . (3i - 2j + k) = 3 + 4 + 3 = 10$

$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$

Let θ be angle between $\vec{a}$ & $\vec{b}$ then $\cos\theta = \dfrac{\vec{a} . \vec{b}}{|\vec{a}|\,|\vec{b}|} = \dfrac{10}{\sqrt{14}\sqrt{14}} = \dfrac{10}{14}$

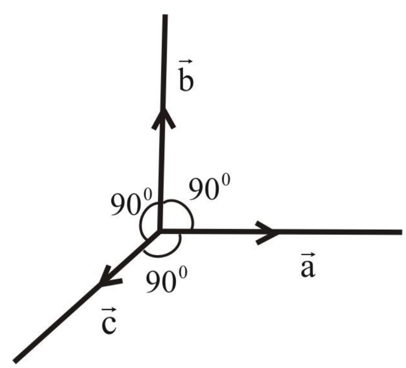
$\cos\theta = \dfrac{5}{7}$ $\theta = \cos^{-1}\dfrac{7}{5}$

6) Show that the vectors $\frac{1}{7}(2i + 3j + 6k)$, $\frac{1}{7}(3i - 6j + 2k)$ and $\frac{1}{7}(6i + 2j + 3k)$ are mutually perpendicular.

Let $\vec{a} = \frac{1}{7}(2i + 3j + 6k)$

$\vec{b} = \frac{1}{7}(3i - 6j + 2k)$

$\vec{c} = \frac{1}{7}(6i + 2j + 3k)$



consider $\vec{a} . \vec{b} = \frac{1}{7}(2i + 3j + 6k) . \frac{1}{7}(3i - 6j + 2k)$

$= \frac{1}{49}\{(2i + 3j + 6k) . (3i - 6j + 2k)\}$

$= \frac{1}{49}\{6 - 18 + 12\} = \frac{1}{49}\{0\}$

$\therefore \vec{a} . \vec{b} = 0$ $\therefore \vec{a}$ is perpendicular to $\vec{b}$

///ly We can show that $\vec{b} . \vec{c} = 0$ & $\vec{c} . \vec{a} = 0$

$\therefore \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors.

7) If $\vec{a} = 5i - j - 3k,\ \vec{b} = i + 3j - 5k$ then show that the vectors $\vec{a} + \vec{b}$ & $\vec{a} - \vec{b}$ are perpendicular.

$\vec{a} + \vec{b} = 5i - j - 3k + i + 3j - 5k = 6i + 2j - 8k$

$\vec{a} - \vec{b} = 5i - j - 3k - (i + 3j - 5k) = 4i - 4j + 2k$

consider $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = (6i + 2j - 8k).(4i - 4j + 2k) = 24 - 8 - 16 = 0$

$\therefore (\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0 \qquad \therefore \vec{a} + \vec{b}$ is perpendicular to $\vec{a} - \vec{b}$

8) If $\vec{a} = 2i + 2j + 3k,\ \vec{b} = -i + 2j + k$ and $\vec{c} = 3i + j$ and such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$ then find λ.

$\vec{a} + \lambda\vec{b} = 2i + 2j + 3k + \lambda(-i + 2j + k)$

$\therefore \vec{a} + \lambda\vec{b} = (2 - \lambda)i + (2 + 2\lambda)j + (3 + \lambda)k$

Given $\vec{a} + \lambda\vec{b}$ is $\perp^{r}$ $\vec{c}$.

$(\vec{a} + \lambda\vec{b}).\ \vec{c} = 0$

$\{(2 - \lambda)i + (2 + 2\lambda)j + (3 + \lambda)k\}.\ \{3i + j\} = 0$

$6 - 3\lambda + 2 + 2\lambda = 0 \Rightarrow -\lambda + 8 = 0 \qquad \therefore \lambda = 8$

9) If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{o}$ then find $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$

$|\vec{a}| = 1,\ |\vec{b}| = 1,\ |\vec{c}| = 1$

$\vec{a}.(\vec{a} + \vec{b} + \vec{c}) = \vec{a}.\vec{o}$

$\vec{a}^2 + \vec{a}.\vec{b} + \vec{a}.\vec{c} = 0$

$\vec{a}.\vec{b} + \vec{a}.\vec{c} = -\vec{a}^2$

$|||^{ly}\ \vec{b}.\vec{c} + \vec{c}.\vec{a} = -\vec{b}^2$

$\vec{c}.\vec{a} + \vec{a}.\vec{b} = -\vec{c}^2$

Adding $2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = -\vec{a}^2 - \vec{b}^2 - \vec{c}^2 = -1^2 - 1^2 - 1^2$

$2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = -3$

$\because \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = -3/2$

10) Find a vector and unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$

where $\vec{a} = 3i + 2j + 2k$ & $\vec{b} = i + 2j - 2k$

$\vec{a}+\vec{b} = 3i + 2j + 2k + i + 2j - 2k$

$\therefore \vec{a}+\vec{b} = 4i + 4j$

$\vec{a}-\vec{b} = 3i + 2j + 2k - i - 2j + 2k$

$\therefore \vec{a}-\vec{b} = 2i + 4k$

Let $\vec{c}$ be the vector perpendicular to $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ then $\vec{c} = (\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})$

$$\begin{matrix} + & - & + \end{matrix}$$

$$\vec{c} = \begin{vmatrix} i & j & k \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = i(16-0) - j(16-0) + k(0-8)$$

$\therefore \vec{c} = 16i - 16j - 8k$

$|\vec{c}| = \sqrt{16^2 + 16^2 + (-8)^2} = \sqrt{256 + 256 + 64} = \sqrt{576} = 24$

Let $\hat{c}$ be the unit vector perpendicular to $(\vec{a}+\vec{b})$ and $\vec{a}-\vec{b}$

then $\hat{c} = \dfrac{\vec{c}}{|\vec{c}|}$

$$\hat{c} = \frac{16i - 16j - 8k}{\sqrt{24}} = \frac{2i}{3} - \frac{2j}{3} - \frac{k}{3}$$

11) If a unit vector $\vec{a}$ makes angles $\pi/3$ with i, $\pi/4$ with j and and acute angle θ with k then find θ and hence components of $\vec{a}$.

Let α, β, γ be the angles made by $\vec{a}$ with i, j, k then

$\alpha = \pi/3,\ \beta = \pi/4, \gamma = \theta$

Let $\vec{a} = a_1 i + a_2 j + a_3 k$

Given $|\vec{a}| = 1$

then $\cos\alpha = \dfrac{a_1}{|a_1|}$

$\cos\pi/3 = \dfrac{a_1}{1}$

$\therefore a_1 = 1/2$

$\cos\beta = \dfrac{a_2}{|\vec{a}|}$

$\cos\pi/4 = \dfrac{a_2}{1}$

$\therefore a_2 = 1/\sqrt{2}$

$\cos\gamma = \dfrac{a_3}{|\vec{a}|}$

$\cos\theta = a_3$

$a_3 = \cos\theta$

$$\therefore \vec{a} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j + \cos\theta\, k$$

$$|\vec{a}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (\cos\theta)^2}$$

$$1 = \sqrt{\frac{1}{4} + \frac{1}{2} + \cos^2\theta}$$

$$\therefore 1 = \frac{3}{4} + \cos^2\theta$$

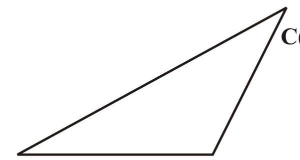
$\cos^2\theta = 1/4$

$\cos\theta = \pm 1/2$

$\theta = 60^0\ or\ 120^0$

$$\therefore \vec{a} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j + \frac{1}{2}k \qquad or \qquad \vec{a} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j - \frac{1}{2}k$$

12) Find the area of triangle with vertices A$(1,1,2)$, B$(2,3,5)$, $C(1,5,5)$



$A \equiv (1,1,2)$ $\quad\overrightarrow{OA} = i + j + 2k$

$B \equiv (2,3,5)$ $\quad\overrightarrow{OB} = 2i + 3j + 5k$

$C \equiv (1,5,5)$ $\quad\overrightarrow{OC} = i + 5j + 5k$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 2i + 3j + 5k - i - j - 2k$

$\overrightarrow{AB} = i + 2j + 3k$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= i + 5j + 5k - i - j - 2k$

$\overrightarrow{AC} = 4j + 3k$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$$= i(6-12) - j(3-0) + k(4-0) = -6i - 3j + 4k$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{51}$$

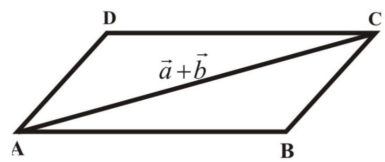
Area of triangle ABC $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{51}$ sq units

13) Find the area of parallelogram whose adjecent sides are $2i - 4j + 5k$ and $i - 2j - 3k$. Also find unit vector parallel to its diagonal.

$\vec{a} = 2i - 4j + 5k \qquad \vec{b} = i - 2j - 3k$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix} = i\{12 + 10\} - j\{-6 - 5\} + k\{-4 + 4\} = 22i + 11j$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{(22)^2 + (11)^2 + 0} = \sqrt{242 + 121 + 0} = \sqrt{363}$$



Area of parallelogram ABCD $= \left|\vec{a} \times \vec{b}\right| = \sqrt{363}$ *sq.units*

Diagonal $\overrightarrow{AC} = \vec{a} + \vec{b}$

$= 2i - 4j + 5k + i - 2j - 3k$

$\overrightarrow{AC} = 3i - 6j + 2k$

$$\left|\overrightarrow{AC}\right| = \sqrt{3^2 + (-6)^2 + 2^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

Unit vector along $\overrightarrow{AC}$, $\qquad \hat{AC} = \dfrac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|} = \dfrac{3i - 6j + 2k}{7}$

Unit vector parallel to $\overrightarrow{AC}$ = Unit vector along $\overrightarrow{AC} = \dfrac{3i - 6j + 2k}{7}$

14) Show that the points A, B, C with position vectors $\vec{a} = 3i - 4j - 4k$, $\vec{b} = 2i - j + k$ & $\vec{c} = i - 3j - 5k$ respectively form the vertices of a right angled triangle.

Given $A \equiv (3, -4, -4) \quad B \equiv (2, -1, 1) \quad C \equiv (1, -3, -5)$

Given $\overrightarrow{OA} = 3i - 4j - 4k$

$\overrightarrow{OB} = 2i - j + k$

$\overrightarrow{OC} = i - 3j - 5k$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 2i - j + k - 3i + 4j + 4k$

$\therefore \overrightarrow{AB} = -i + 3j + 5k$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= i - 3j - 5k - 2i + j - k$

$\therefore \overrightarrow{BC} = -i - 2j - 6k$

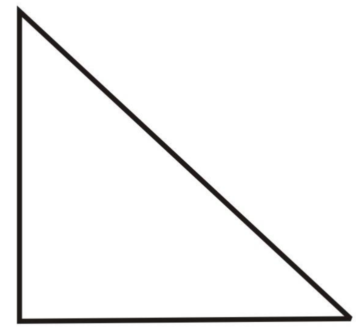
$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= 3i - 4j - 4k - i + 3j + 5k = 2i - j + k$

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = -i + 3j + 5k - i - 2j - 6k + 2i - j + k$

$\therefore \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{O}$

$\therefore \Delta^{le}$ *law is satisfied*



$\therefore \vec{a}, \vec{b}, \vec{c}$ *form a* Δ^{le}

$\left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{35}$ $\quad \left|\overrightarrow{AB}\right|^2 = 35$

$\left|\overrightarrow{BC}\right| = \sqrt{(-1)^2 + (2)^2 + 6^2} = \sqrt{41}$ $\quad \left|\overrightarrow{BC}\right|^2 = 41$

$\left|\overrightarrow{CA}\right| = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6}$ $\quad \left|\overrightarrow{CA}\right|^2 = 6$

We can see that $\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 = 35 + 6 = 41 = \left|\overrightarrow{BC}\right|^2$

$\therefore \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{BC}\right|^2$

$\therefore$ Pythagrous theorem is satified.

i.e. $\vec{a}, \vec{b}, \vec{c}$ form a right angled triangle

14) Prove that $\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = 2\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

$$\text{LHS} = \left(\vec{a}+\vec{b}\right).\left\{\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right\}$$
$$= \left(\vec{a}+\vec{b}\right).\left\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\right\}$$
$$= \left(\vec{a}+\vec{b}\right).\left\{\vec{b}\times\vec{c}-\vec{a}\times\vec{b}+\vec{c}\times\vec{a}\right\}$$
$$= \vec{a}.\left(\vec{b}\times\vec{c}\right)-\vec{a}.\left(\vec{a}\times\vec{b}\right)+\vec{a}.\left(\vec{c}\times\vec{a}\right)+\vec{b}\left(\vec{b}\times\vec{c}\right)-\vec{b}\left(\vec{a}\times\vec{b}\right)+\vec{b}.\left(\vec{c}\times\vec{a}\right)$$
$$= \vec{a}.\left(\vec{b}\times\vec{c}\right)-0+0+0-0+\vec{b}.\left(\vec{c}\times\vec{a}\right)$$
$$= \vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{a}\left(\vec{b}\times\vec{c}\right) = 2\vec{a}.\left(\vec{b}\times\vec{c}\right) = 2\left[\vec{a},\vec{b},\vec{c}\right]$$

14) Show that the vectors $4i - j + k, 3i - 2j - k$ and $i + j + 2k$ are coplanar

Let $\vec{a} = 4i - j + k$

$\vec{b} = 3i - 2j - k$

$\vec{c} = i + j + 2k$

$$\text{consider } \vec{a}.\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} 4 & -1 & 1 \\ 3 & -2 & -1 \\ 1 & 1 & 2 \end{vmatrix}$$
$$= 4\{-4+1\}+1\{6+1\}+1\{3+2\}$$
$$= -12+7+5$$
$$\therefore \vec{a}.\left(\vec{b}\times\vec{c}\right) = 0$$

$\therefore$ vectors are coplanar

Scalar Triple Product of vectors

If $\vec{a}, \vec{b}, \vec{c}$ are non zero vectors then $\vec{a}.\left(\vec{b}\times\vec{c}\right)$ *or* $\left(\vec{a}\times\vec{b}\right).\vec{c}$ is called scalar triple product of vectors.

The scalar triple product of $\vec{a}, \vec{b}, \vec{c}$ is denoted by $\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$ or $\left[\vec{a}, \vec{b}, \vec{c}\right]$

$$\text{Then } \vec{a}.\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

since scalar triple product is a determinent , all determinent properties are satisfied.

Properties

1) $\vec{a}.\left(\vec{b}\times\vec{c}\right)$ is a scalar.

2) $\vec{a}.\left(\vec{b}\times\vec{c}\right)=\vec{b}.\left(\vec{c}\times\vec{a}\right)=\vec{c}.\left(\vec{a}\times\vec{b}\right)$

$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\left[\vec{b}\;\vec{c}\;\vec{a}\right]=\left[\vec{c}\,\vec{a}\,\vec{b}\right]$

3) $\vec{a}.\left(\vec{a}\times\vec{b}\right)=0$, $\quad \vec{a}.\left(\vec{b}\times\vec{b}\right)=0$

$\left[\vec{a}\,\vec{a}\;\vec{b}\right]=0 \qquad \left[\vec{a}\;\vec{b}\;\;\vec{b}\right]=0$

4) $\vec{a}.\left(\vec{b}\times\vec{c}\right)=-\,\vec{a}.\left(\vec{c}\times\vec{b}\right)$

$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=-\left[\vec{a}\;\vec{c}\;\;\vec{b}\right]$ *etc.*

5) Dot and cross can be interchanged

$\vec{a}.\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right).\vec{c}$

$\left(\vec{b}\times\vec{c}\right).\vec{a}=\vec{b}.\left(\vec{c}\,.\vec{a}\right)$ *etc.*

Coplanar vector:

The vectors are said to be coplanar if they lie on same plane or parallel planes.

The condition for the vectors to be coplanar is $\vec{a}.\left(\vec{b}\times\vec{c}\right)=0$

Problems:

1) Find the scalar triple product of vectors $i+2j+3k$, $-i-j+k$ and $i+j+k$

$$\text{Scalar triple product}=\begin{vmatrix}1 & 2 & 3\\ -1 & -1 & 1\\ 1 & 1 & 1\end{vmatrix}$$

$$=1\left(-1-1\right)-2\left(-1-1\right)+3\left(-1+1\right)$$

$$=-2+4+0=2$$

2) Prove that $\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right]=2\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

$$\text{LHS}=\left(\vec{a}+\vec{b}\right).\left\{\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right\}$$
$$=\left(\vec{a}+\vec{b}\right).\left\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\right\}$$
$$=\left(\vec{a}+\vec{b}\right).\left\{\vec{b}\times\vec{c}-\vec{a}\times\vec{b}+\vec{c}\times\vec{a}\right\}$$
$$=\vec{a}.\left(\vec{b}\times\vec{c}\right)-\vec{a}.\left(\vec{a}\times\vec{b}\right)+\vec{a}.\left(\vec{c}\times\vec{a}\right)+\vec{b}\left(\vec{b}\times\vec{c}\right)-\vec{b}\left(\vec{a}\times\vec{b}\right)+\vec{b}.\left(\vec{c}\times\vec{a}\right)$$
$$=\vec{a}.\left(\vec{b}\times\vec{c}\right)-0+0+0-0+\vec{b}.\left(\vec{c}\times\vec{a}\right)$$
$$=\vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{a}\left(\vec{b}\times\vec{c}\right)=2\vec{a}.\left(\vec{b}\times\vec{c}\right)=2\left[\vec{a},\vec{b},\vec{c}\right]$$

3) Show that the vectors $4i - j + k,\ 3i - 2j - k$ and $i + j + 2k$ are coplanar

Let $\vec{a} = 4i - j + k$

$\vec{b} = 3i - 2j - k$

$\vec{c} = i + j + 2k$

$$\text{consider } \vec{a}.\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix} 4 & -1 & 1 \\ 3 & -2 & -1 \\ 1 & 1 & 2 \end{vmatrix}$$
$$=4\{-4+1\}+1\{6+1\}+1\{3+2\}$$
$$=-12+7+5$$
$$\therefore \vec{a}.\left(\vec{b}\times\vec{c}\right)=0$$

$\therefore$ vectors are coplanar

4) Find λ if the vectors $i + j + 2k,\ \lambda i - j + k\ \&\ 3i - 2j - k$ are coplanar.

Given that vectors are coplanar

$$\therefore \begin{vmatrix} 1 & 1 & 2 \\ \lambda & -1 & 1 \\ 3 & -2 & -1 \end{vmatrix} = 0$$

$$= 1(1+2) - 1(-\lambda - 3) + 2(-2\lambda + 3) = 0$$

$$= 3 + \lambda + 3 - 4\lambda + 6 = 0$$

$$= -3\lambda + 12 = 0$$

$$\therefore \lambda = 4$$